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1984 J. Phys. A: Math. Gen. 17 2899

(http://iopscience.iop.org/0305-4470/17/15/009)

J. Phys. A: Math. Gen. 17 (1984) 2899–2926. Printed in Great Britain

Dual bases and some coupling coefficients for $SU_4 \supset SU_2 \times SU_2$, $SU_n \supset SO_n$ and $Sp_4 \supset U_2$

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Received 11 January 1984, in final form 5 June 1984

Abstract. General analytical expressions for isoscalar factors (reduced Wigner coefficients) for the couplings $(p\dot{0})\times(\dot{0}q)$ to $(\lambda\dot{0}\mu)$ and $(p_1\dot{0})\times(p_2\dot{0})$ to $(\lambda\nu\dot{0})$ in the chains $SU_4 \supset SU_2 \times SU_2$ and $SU_n \supset SO_n$ $(n \geqslant 3)$ are derived by using the algebra of complementary groups and methods of analytical continuation. A review of alternative approaches to the problem grounded on the concept of biorthogonal systems of non-canonical bases is given. It is demonstrated that the construction of dual bases (polynomial and stretched or related ones) is associated with the use of certain bilinear combinations of isoscalar factors and solutions of the boundary value problem for isoscalar factors. Overlaps are given for different non-canonical basis states of the two-parametric irreducible representations for the chains $SU_4 \supset SU_2 \times SU_2$, $SU_n \supset SO_n$ and $Sp_4 \supset U_2$, as well as the expansions of the projected basis states for $Sp_4 \supset U_2$ in terms of canonical basis states. A new expression for some special SU_2 Clebsch–Gordan coefficients is also given.

1. Introduction

The Clebsch–Gordan or Wigner coefficients of unitary groups for non-canonical bases, corresponding to the reductions $SU_4 \supset SU_2 \times SU_2$, $SU_n \supset SO_n$ and, especially, $SU_3 \supset SO_3$, are very important for different group theoretical models of the nucleus. These chains of subgroups have applications in Wigner supermultiplet theory and in the theory of collective excitations; $Sp_4 \supset U_2$ also has application in the theory of the five-dimensional quasispin. All these restrictions are not multiplicity free and analytic methods allow us to construct only non-orthogonal basis states and coupling coefficients.

An extensive literature exists on the $SU_3 \supset SO_3$ problems, beginning with the papers by Elliott (1958), Bargmann and Moshinsky (1960, 1961), Sharp *et al* (1969) and von Baeyer and Sharp (1970), where projected, polynomial and stretched non-canonical bases were introduced. All these bases are, however, non-orthogonal. Asherova and Smirnov (1970, 1973) (cf Filippov *et al* 1981) have very effectively used projection operators expressed as polynomials in the SO_3 generators instead of integral operators in the group elements. The review by Moshinsky *et al* (1975) was a serious attempt to discuss systematically the different bases of $SU_3 \supset SO_3$ from a uniform viewpoint, but only Ališauskas (1978) demonstrated that the renormalised basis of Bargmann–Moshinsky (1960, 1961) and stretched basis of Sharp *et al* (1969) were dual, i.e. that they form a biorthogonal system of vectors. It was shown by Ališauskas (1978, see Ališauskas *et al* 1981) that the renormalised Bargmann–Moshinsky states can be expressed as certain linear combinations of orthonormal $SU_3 \supset SO_3$ states with special isoscalar factors (reduced Wigner coefficients or, briefly, isofactors) as expansion coefficients (the explicit expressions of these isofactors are not necessary). Otherwise, the Bargmann–Moshinsky states can be obtained by means of a coupling procedure in which special bilinear combinations of isofactors are used as the coupling coefficients for direct product states as defined by Moshinsky *et al* (1975). The stretched states can be obtained by means of the coupling procedure with the help of isofactors which satisfy special boundary conditions. The matrices formed by the former coupling coefficients and by the last ones are mutually inverse. The linearly dependent generalised Bargmann–Moshinsky states may be expanded in terms of the complete system, the expansion coefficients being the isofactors which couple to the stretched states. The concept of dual bases has permitted us to construct the invariants of the group and to find the expansions of given functions in terms of the desirable non-orthogonal basis states. The expressions of the metric tensors and overlaps for different above-mentioned bases as well as the mutual expansions of the different non-canonical and canonical basis states were considerably simplified by Ališauskas (1978).

A polynomial supermultiplet basis for $SU_4 \supset SU_2 \times SU_2$ was proposed by Brunet and Resnikoff (1970). The basis taken for the polynomial by Ahmed and Sharp (1972) in fact is a generalisation of the stretched basis, because it can be expanded by means of triangular matrices (given in explicit form by Ališauskas and Norvaišas (1979)) in terms of the projected basis states introduced by Draayer (1970) (cf Ahmed and Sharp 1972, Norvaišas and Ališauskas 1977, Norvaišas 1981).

The basis states of the symmetric (one-row) representations of $U_n \supset O_n \supset O_{n'} \dot{+} O_{n''}$ ($n'+n''=n$, $n'' \geqslant 1$) were obtained by Ališauskas and Vanagas (1972) by means of the complementary group technique. There were some attempts (Prasad 1972, Mickelsson 1973, Jarvis 1974) to construct more general $SU_n \supset SO_n$ states before Ališauskas and Norvaišas (1980) found the biorthogonal systems of isofactors for couplings of the basis states of two symmetric covariant irreps (irreducible representations) as well as of covariant and contravariant irreps in the chains $SU_4 \supset SU_2 \times SU_2$ and $SU_n \supset SO_n$†. The methods used for $SU_4 \supset SU_2 \times SU_2$ have been generalised by means of the complementary group technique (Moshinsky and Quesne 1970, 1971, Quesne 1973, Deenen and Quesne 1983, Knyr *et al* 1975, Ališauskas 1974, 1976, 1983c) in order to solve completely the labelling problem for the states of two-parametric (covariant and mixed tensor) irreps in the $SU_n \supset SO_n$ chain. However, most of the expressions for isofactors given by Ališauskas and Norvaišas (1980) are far from being optimal. Later (Ališauskas 1982, 1983a) the corresponding isofactors for coupling states of two symmetric irreps to projected basis states (Draayer states), as well as the overlap matrices, have been expressed in terms of triple sums.

It was shown by Ališauskas and Norvaišas (1980) that the states of projected and stretched bases of the two-parametric mixed tensor irreps of $SU_4 \supset SU_2 \times SU_2$ are proportional. (Analogous $SU_3 \supset SO_3$ bases are related to each other by triangular matrices—see Sharp *et al* (1969), Moshinsky *et al* (1975), Ališauskas (1978).) In such a way the solution of the boundary value problem for isofactors which perform the couplings $(p\dot{0}) \times (\dot{0}q)$ to $(\lambda\dot{0}\mu)$ in chains $SU_n \supset SO_n$ was proposed‡ (Ališauskas 1983a).

† Some other examples of biorthogonal systems associated with the internal or external multiplicity problem of irreps are discussed in the review by Ališauskas (1983c).

‡ It is convenient for our purposes to denote the irreps of SU_n by differences of the lengths of rows in Young tableaux. The irreps of SO_n, however, are denoted by Young tableaux as $[L_1 L_2 \dot{0}]$. Only two-parametric irreps are under consideration here. The necessary number of zeros is denoted by $\dot{0}$.

However, the expressions obtained by Ališauskas (1982, 1983a) are not sufficiently convenient in some domains of the parameters. For example, they do not simplify considerably when scalars of SO_n in irreps $(\lambda \dot{0} \mu)$ or $(\lambda \nu \dot{0})$ of SU_n appear, contrary to the results obtained by Hecht and Suzuki (1983) for $SU_3 \supset SO_3$.

The main result of this paper is a new expression for isofactors which couple the states of irreps $(p\dot{0})$ and $(\dot{0}q)$ to the states of a polynomial basis of $SU_4 \supset SU_2 \times SU_2$ and $SU_n \supset SO_n$. The combination of methods of Ališauskas and Norvaišas (1980) and those of Hecht and Suzuki (1983) with complementary group technique allowed us to express these quantities as fourfold sums.

The analytical continuation of the quantity obtained by means of special substitutions of the parameters of irreps (see Ališauskas and Jucys 1967, 1974) permitted us to find the isofactors for coupling $(p_1\dot{0}) \times (p_2\dot{0})$ to $(\lambda \nu \dot{0})$ as well. In such a way the bilinear combinations of isofactors (summed over the multiplicity label) were found. In order to get a general view of the solutions of the problem and, in particular, to demonstrate the completeness of results for the labelling problem, some results of Ališauskas and Norvaišas (1980) and Ališauskas (1982, 1983a) are discussed and developed for the solution of the boundary value problem.

The analytical continuation of the isoscalar factors under consideration allows us to obtain new information about the basis states of the complementary group Sp_4, and in particular to find a new expansion for projected basis states of five-dimensional quasispin by Smirnov and Tolstoy (1973) (cf Ališauskas 1983b) in terms of canonical basis states, as well as the corresponding overlaps.

All the expressions for isofactors and overlaps are derived at first for $SU_4 \supset SU_2 \times SU_2$, but for economy of space they are written below only for $SU_n \supset SO_n$ (with slight difference in phases in the $SU_3 \supset SO_3$ case). Some details of the analysis are given in appendices 1 and 2.

The new expression (in appendix 2) for the SU_2 Clebsch–Gordan coefficients with a zero-valued projection of angular momentum may be of some importance for angular momentum theory.

The methods of Ališauskas (1978), effective only in the $SU_3 \supset SO_3$ case, allowed us to obtain the overlaps of the non-canonical basis states given in appendix 3. Some special isofactors and overlaps are transformed into their most convenient forms in appendix 4.

The results of this paper are useful for SU_3, SU_4 and SU_n irreducible decomposition of the nuclear Hamiltonian (Taurinskas and Vanagas 1977) and the nuclear density matrix (Kalinauskas and Svičiulis 1982) with rotationally invariant interaction (cf Hecht and Suzuki 1983) and vector and tensor forces included.

The analytical continuation of discrete functions under consideration is possible when the intervals of summation are fixed, i.e. when these functions may be expressed as polynomials in the remaining parameters. In the expressions which follow, the intervals of summation parameters are restricted by values for which simple factorials in denominators are finite. There is no problem with the analytical continuation in cases when the remaining factorials (in fact the ratios of factorials) are expressed as quasipowers or Pochhammer symbols with fixed indices. However, the use of simple and double factorials is more convenient. For n even, almost all double factorials can be changed to simple factorials. We will use the convention that double factorials of odd negative integers are always finite (for example $(-1)!! = 1$), while for even negative integers they are infinite. The indeterminacies of type 0^0 and some others mentioned below are taken as 1.

If whole Clebsch–Gordan (Wigner) coefficients for the chain $SU_n \supset SO_n \supset SO_{n-1} \supset SO_{n-2} \ldots$ are needed, the results of Norvaišas and Ališauskas (1974) (the expressions for special isofactors of $SO_n \supset SO_{n-1}$) should be taken into account (with the corresponding phase changes discussed by Ališauskas and Norvaišas 1980).

2. Complementary groups, polynomial states for mixed tensor irreps $(\lambda\dot{0}\mu)$ and bilinear combinations of isofactors for coupling $(p\dot{0})\times(\dot{0}q)$ to $(\lambda\dot{0}\mu)$

Let us take $2n$ boson creation operators η_i and ξ_i which transform with respect both to SO_n as two vectors and to SU_n as a covariant and a contravariant vector, respectively. The corresponding annihilation operators $\bar{\eta}_j$ and $\bar{\xi}_j$ transform as a contravariant and a covariant vector of SU_n, respectively. These operators satisfy the usual commutation relations

$$\begin{aligned}
&[\eta_i, \xi_j] = [\bar{\eta}_i, \xi_j] = [\eta_i, \bar{\xi}_j] = [\bar{\eta}_i, \bar{\xi}_j] = [\eta_i, \eta_j] \\
&\qquad = [\bar{\eta}_i, \bar{\eta}_j] = [\xi_i, \xi_j] = [\bar{\xi}_i, \bar{\xi}_j] = 0, \\
&[\bar{\eta}_i, \eta_j] = [\bar{\xi}_i, \xi_j] = \delta_{ij}.
\end{aligned} \tag{2.1}$$

Then the elementary scalars of SO_n

$$\begin{aligned}
&(\eta\xi) = \sum_i \eta_i\xi_i, \quad (\eta\eta), \quad (\xi\xi), \quad (\bar{\eta}\bar{\xi}), \quad (\bar{\eta}\bar{\eta}), \quad (\bar{\xi}\bar{\xi}), \\
&N_\eta = \sum_i \eta_i\bar{\eta}_i, \quad N_\xi, \qquad E_{\eta\xi} = \sum_i \eta_i\bar{\xi}_i, \quad E_{\xi\eta}
\end{aligned} \tag{2.2}$$

form the Lie algebra of the complementary group Sp(2, 2). The elementary scalars of SU_n $(\eta\xi)$, $(\bar{\eta}\bar{\xi})$ belong to the Lie algebra of SU(1, 1).

The commutation relations between the elementary scalars (2.2) may be easily written. However, spherical coordinate are more useful for our purposes than Cartesian ones (cf Ališauskas and Vanagas 1972, Deenen and Quesne 1983, § 3). Then elementary scalars (2.2) can be obtained by the coupling procedure. The coupling procedure allows one to construct the elementary highest weight states for the irrep [1, 1] of SO_n (or O_3 if $n = 3$) as well. Let us denote them as

$$[\eta\xi]_{1,1} = \eta_{(+1)}\xi_{(+2)} - \eta_{(+2)}\xi_{(+1)} \tag{2.3a}$$

($\eta_{(\pm k)}$, $\xi_{(\pm k)}$ are the spherical components),

$$[\eta\xi]_1 = \eta_1\xi_0 - \eta_0\xi_1, \tag{2.3b}$$

$$[\eta\xi]_{1,0} = \eta_{1/2\ 1/2}\xi_{1/2-1/2} - \eta_{1/2-1/2}\xi_{1/2\ 1/2} \tag{2.3c}$$

for the special cases of $SO_3(O_3)$ and $SU_2\times SU_2 \sim SO_4$.

Thus all the elementary permissible diagrams (EPD) are constructed. The complementary group technique (cf Ališauskas and Vanagas 1972) allows one to obtain normalised highest weight states for any two-parametric irrep of SO_n as polynomials in $(\eta\eta)$, $(\xi\xi)$, $[\eta\xi]_{1,1}$, η_{hw}, ξ_{hw} (in particular, the power of $[\eta\xi]_1$ is not higher than the first in the SO_3 case). Such states belong to special direct product states of SU_n for the representation $(\lambda\dot{0})\times(\dot{0}\mu)$. In order to eliminate the unwanted irreps of SU_n, one must act rather with the projection operator of the complementary group SU(1, 1) than with the SU_n Casimir operator used (for $n = 3$) in an analogous situation by

Hecht and Suzuki (1983) in the case of SO_3 scalar states. The projection operator (Ališauskas and Norvaišas 1980)

$$P^a_{r,r'} = \frac{(a+n-1)}{[r!r'!(a+r+n-1)![a+r'+n-1)!]^{1/2}} \times \sum_x \frac{(-1)^x}{x!}(a+n-2-x)!(\eta\xi)^{r+x}(\bar{\eta}\bar{\xi})^{r'+x} \tag{2.4}$$

leaves only the state of the irrep $(\lambda\dot{0}\mu)$ of SU_n $(\lambda+\mu=a)$ when applied in the direct product state $(\lambda+r', \dot{0})\times(\dot{0}, \mu+r')$ and transforms it into a homogeneous polynomial of degrees $\lambda+r$ and $\mu+r$ in η_i and ξ_i, respectively.

The relation by Asherova *et al* (1980)

$$E^b_\beta E^a_\alpha = \sum_{i,t} \frac{a!b!(N_{\beta,\alpha})^i(N_{\beta,\alpha+\beta})^t}{(a-i)!(b-i-t)!(i-t)!t!2^t} E^{a-i}_\alpha E^{i-t}_{\alpha+\beta} E^t_{\alpha+2\beta} E^{b-i-t}_\beta \tag{2.5}$$

($N_{\beta,\alpha}$, $N_{\beta,\alpha+\beta}$ are the structure constants of the Lie algebra) allows one to transfer the annihilation operators through the creation operators and by means of some summation formulae of binomial coefficients (Jucys and Bandzaitis 1977) to obtain the expansion for $SU_4 \supset SU_2 \times SU_2$ or $SU_n \supset SO_n$ irreducible states as polynomials in the above-mentioned EPD and $(\eta\xi)$. In fact the states obtained are linear combinations of the orthonormal states. The weight factors are special isofactors, which correspond to the contribution of the state of the irrep $(\lambda\dot{0}\mu)$ in the direct product state.

Now we can use the polynomials obtained, following Hecht and Suzuki (1983), as the generating functions for isofactors. First of all let us expand the particular case of the polynomial (more exactly, monomial) with disappearing elementary scalars $(\eta\eta)$ and $(\xi\xi)$ in terms of direct product states. The special isofactors needed may be expressed as overlaps of the special state under consideration and special states of the symmetric irreps $(p\dot{0})$ and $(\dot{0}q)$ (cf Ahmed and Sharp 1972) and can be found by the usual methods of second quantisation. (The states must be expanded as polynomials in the creation operators.)

Let us replace the monomials in $(\eta\xi)$, $[\eta\xi]_{1,1}$, η_{hw}, ξ_{hw}, included in the general polynomial states for $SU_4 \supset SU_2 \times SU_2$ or $SU_n \supset SO_n$, by the expansions obtained. The action with the operators $(\eta\eta)^a$ and $(\xi\xi)^b$ allows us to represent the general polynomial state as an expansion in direct product states.

The expansion coefficient contains a fourfold sum. Two sums belong to the ${}_3F_2(1)$ class and may be transformed into a more convenient form by the methods of Jucys and Bandzaitis (1977) used for SU_2 Clebsch–Gordan coefficients.

The above-discussed deduction was simplest in the $SU_4 \supset SU_2 \times SU_2$ case. The substitutions (Ališauskas and Norvaišas 1980)

$$p \to p + \tfrac{1}{2}n - 2, \qquad q \to q + \tfrac{1}{2}n - 2, \qquad \lambda \to \lambda + \tfrac{1}{2}n - 2, \qquad \mu \to \mu + \tfrac{1}{2}n - 2, \tag{2.6}$$

$$\begin{aligned} &S \to \tfrac{1}{2}(L_1 + L_2 + n) - 2, \qquad T \to \tfrac{1}{2}(L_1 - L_2), \qquad j_1 \to \tfrac{1}{2}l_1 + \tfrac{1}{4}n - 1, \\ &j_2 \to \tfrac{1}{2}l_2 + \tfrac{1}{4}n - 1, \qquad j_{10} \to \tfrac{1}{2}l_{10} + \tfrac{1}{4}n - 1, \qquad j_{20} \to \tfrac{1}{2}l_{20} + \tfrac{1}{4}n - 1 \end{aligned} \tag{2.7}$$

(the symbols S, T, j_1, j_2 are used here for parameters of spin or isospin type), grounded on the dependence between the coefficients of the Wigner–Racah calculus of complementary groups, lead to the following expression for bilinear combinations of

$SU_n \supset SO_n$ isoscalar factors:

$$\begin{bmatrix} (p\dot{0}) & (\dot{0}q) & (\lambda\dot{0}\mu)_B \\ l_1 & l_2 & (l_{10}l_{20})[L_1L_2] \end{bmatrix}$$

$$= \sum_\omega \begin{bmatrix} (p\dot{0}) & (\dot{0}q) & (\lambda\dot{0}\mu) \\ l_1 & l_2 & \omega[L_1L_2] \end{bmatrix} \begin{bmatrix} (\lambda\dot{0}) & (\dot{0}\mu) & (\lambda\dot{0}\mu) \\ l_{10} & l_{20} & \omega[L_1L_2] \end{bmatrix}$$

$$= \left(\frac{(\lambda+\mu+n-1)(2l_1+n-2)(2l_2+n-2)(2l_{10}+n-2)!!}{(\lambda+\mu+n-2)!(p-\lambda)!(\lambda+q+n-1)!(2L_1+n-2)!!}\right.$$

$$\left.\times\frac{(2l_{20}+n-2)!!(L_1-l_{20})!(l_{20}-L_2)!}{(2L_2+n-4)!!(L_1+L_2+n-3)!}\right)^{1/2}$$

$$\times\frac{(\lambda+\mu-L_1+L_2+n-2)!!}{2^{(\lambda+\mu+L_1-L_2+n-3)/2}}$$

$$\times\frac{\nabla_{n[4,7]}(l_1l_2;L_1L_2)\,W_n(p,l_1)\,W_n(q,l_2)}{W'_n(\lambda,l_{10})\,W'_n(\mu,l_{20})}$$

$$\times\sum_{x,y,z,u}\frac{(-1)^{\psi+x+z}2^{x-z-u}(\lambda+\mu+n-2-x)!}{u!z!(y-z)!(x-y-z)!(L_1-l_{20}-x+y+z)!(l_{20}-L_2-y+z)!}$$

$$\times\frac{(p-\lambda+x)!(L_1-l_{20}-x+2y)!}{(\lambda+\mu-L_1+L_2+n-2-2z)!![\frac{1}{2}(p-\lambda+l_{10}-l_1)+y-u]!}$$

$$\times\frac{(l_{20}-L_2+x-2y)!}{[\frac{1}{2}(p-\lambda+l_{20}-l_2)+x-y-u]!}$$

$$\times\frac{(l_1+l_2+L_1-L_2+n-2+2u)!!}{[\frac{1}{2}(l_1+l_{10}-p+\lambda)-L_2-x+y+u]![\frac{1}{2}(l_2+l_{20}-p+\lambda)-L_2-y+u]!}$$

$$\times\{[\tfrac{1}{2}(\lambda-l_{10})-y]![\tfrac{1}{2}(\mu-l_{20})-x+y]!(p-\lambda+l_{10}+l_1+n-2+2y)!!$$

$$\times(p-\lambda+l_{20}+l_2+n-2+2x-2y)!!\}^{-1}. \qquad (2.8)$$

Here ω is a multiplicity label of orthonormal basis states,

$$l_{10}+l_{20}=L_1+L_2, \qquad p-q=\lambda-\mu.$$

Now we have obtained a complete system of non-orthogonal isofactors of $SU_n \supset SO_n$ for the coupling of $(p\dot{0})\times(\dot{0}q)$ to $(\lambda\dot{0}\mu)$ into polynomial (B) states.

Here the following notations are introduced:

$$W_n(p,l)=[(p-l)!!(p+l+n-2)!!]^{1/2}, \qquad (2.9)$$

$$W'_n(p,l)=[(p+l+n-2)!!/(p-l)!!]^{1/2}, \qquad (2.10)$$

$$\nabla_{n[i_1\ldots i_k]}(l_1l_2;L_1L_2)=\left(\prod_{i=0}^{7}A_i(l_1l_2;L_1L_2)\right)^{1/2}\Big/\prod_{i\to[i_1\ldots i_k]}A_i(l_1l_2;L_1L_2), \qquad (2.11)$$

$$\begin{aligned} A_0&=(l_1+l_2+L_1+L_2+2n-6)!!, & A_4&=(l_1+l_2+L_1-L_2+n-2)!!,\\ A_1&=(L_1+L_2-l_1+l_2+n-4)!!, & A_5&=(L_1-L_2-l_1+l_2)!!,\\ A_2&=(l_1-l_2+L_1+L_2+n-4)!!, & A_6&=(l_1-l_2+L_1-L_2)!!,\\ A_3&=(l_1+l_2-L_1+L_2+n-4)!!, & A_7&=(l_1+l_2-L_1-L_2)!!. \end{aligned} \qquad (2.12)$$

In the phase factors we use for $n=3$:

$$\psi=\tfrac{1}{2}(l_1+l_2-L_1-L_2), \qquad \psi'=\tfrac{1}{2}(l_1-l_2-L_1+L_2), \qquad \varphi=\tfrac{1}{2}(p_1-l_1+p_2-l_2),$$
$$\psi=\psi'=\varphi=0 \qquad \text{otherwise.} \tag{2.13}$$

3. Stretched states and the boundary value problem for coupling $(p\dot{0})\times(\dot{0}q)$ to $(\lambda\dot{0}\mu)$

A well known construction (for $SU_3 \supset SO_3$ perhaps starting with the papers by Engeland (1965), Vergados (1968), Asherova and Smirnov (1970), von Baeyer and Sharp (1970)) allowed us (Ališauskas and Norvaišas 1980) to write an expression for isofactors which couple the states of irreps $(p\dot{0})$ and $(\dot{0}q)$ to projected Draayer states of $SU_4 \supset SU_2 \times SU_2$. It appeared that such isofactors (after renormalisation) satisfy elementary boundary conditions, i.e. that they are proportional to the isofactors which perform the coupling to stretched basis states. The stretched (S) basis is dual to the polynomial basis B introduced in § 2 (i.e. the bases S and B are obtained by means of direct and inverse transformations of the orthonormal basis, respectively); the corresponding isofactors satisfy the following boundary conditions:

$$\begin{bmatrix} (p\dot{0}) & (\dot{0}q) & (\lambda\dot{0}\mu)_S \\ l_1 & l_2 & (l_{10}l_{20})[L_1L_2] \end{bmatrix} = \delta_{l_2 l_{20}} \tag{3.1}$$

if $p=\lambda$, $q=\mu$ and the parameters l_1, l_2 take values satisfying the condition $l_1+l_2=L_1+L_2$.

In fact, the solution satisfying boundary condition (3.1) enables us to express an arbitrary isofactor of the class under consideration in terms of its values in a certain region of parameters, equivalent to multiplicity labels. For example, the general bilinear combination of isofactors takes the form

$$\sum_{\omega}\begin{bmatrix} (p'\dot{0}) & (\dot{0}q') & (\lambda\dot{0}\mu) \\ l_1' & l_2' & \omega[L_1L_2] \end{bmatrix}\begin{bmatrix} (p\dot{0}) & (\dot{0}q) & (\lambda\dot{0}\mu) \\ l_1 & l_2 & \omega[L_1L_2] \end{bmatrix}$$
$$=\sum_{l_{20}}\begin{bmatrix} (p_1'\dot{0}) & (\dot{0}q') & (\lambda\dot{0}\mu)_B \\ l_1' & l_2' & (l_{20})[L_1L_2] \end{bmatrix}\begin{bmatrix} (p\dot{0}) & (\dot{0}q) & (\lambda\dot{0}\mu)_S \\ l_1 & l_2 & (l_{20})[L_1L_2] \end{bmatrix}. \tag{3.2}$$

The completeness of the system of isofactors which satisfy the boundary condition (3.1) is evident if their existence is proved. The expansion (3.2) also allows us to prove the completeness of the polynomial bases introduced in § 2.

Between the labels of Draayer states and stretched states there is the following correspondence: $K_S=S$, $K_T=j_{10}-j_{20}$. Some quantities from equation (3.3) of Ališauskas and Norvaišas (1980) were represented as Clebsch–Gordan coefficients of SU_2 and using the Wigner–Racah algebra of SU_2 for summation, the following expression for $SU_4 \supset SU_2 \times SU_2$ isofactors satisfying the boundary conditions was obtained (Ališauskas 1983a) ($j_{10}+j_{20}=S\geqslant T$):

$$\begin{bmatrix} (p00) & (00q) & (\lambda 0\mu)_S \\ j_1j_1 & j_2j_2 & (j_{10}j_{20})ST \end{bmatrix}$$
$$=\left(\frac{(\frac{1}{2}\lambda+j_{10}+1)(\frac{1}{2}\mu+j_{20}+1)(2S+1)}{(2j_{10}+1)!(2j_{20}+1)!}\right.$$

$$\times\frac{(2j_1+1)(2j_2+1)(S-T)!(S+T+1)!(p-\lambda)!(\lambda+\mu+3)!}{[\frac{1}{2}(\lambda+\mu)-S]![\frac{1}{2}(\lambda+\mu)+S+1]!(\lambda+q+3)!}\Bigg)^{1/2}$$

$$\times\sum_{m_1,m_2}(-1)^{\lambda+j_1+S+m_2}\left(\frac{[\frac{1}{2}(\lambda+q)+j_{20}+m_1+1]![\frac{1}{2}(\lambda+q)-j_{20}-m_1+1]!}{[\frac{1}{2}(p-\lambda)-j_{10}+m_1]![\frac{1}{2}(p-\lambda)+j_{10}-m_1]!}\right)^{1/2}$$

$$\times\begin{bmatrix} j_1 & j_2 & T \\ m_1 & m_2 & j_{10}-j_{20}\end{bmatrix}\begin{Bmatrix} j_1 & j_2 & S \\ \frac{1}{4}p-\frac{1}{2}m_1 & \frac{1}{4}q+\frac{1}{2}m_2 & \frac{1}{4}(\lambda+\mu)-\frac{1}{2}S \\ \frac{1}{4}p+\frac{1}{2}m_1 & \frac{1}{4}q-\frac{1}{2}m_2 & \frac{1}{4}(\lambda+\mu)+\frac{1}{2}S\end{Bmatrix}. \tag{3.3}$$

On the RHS of (3.3) both the stretched $9j$-coefficient of SU_2 (denoted by braces) and the Clebsch–Gordan coefficient of SU_2 (denoted by square brackets) appeared. (The phase factor is slightly changed in comparison with Ališauskas (1983a).)

The use of specially chosen (not the most symmetric) expressions for the quantities of angular momentum theory in (3.3) allowed us to simplify the final expression up to the triple sum and to continue it analytically for isofactors of the chain $SU_n \supset SO_n$ ($l_{10}+l_{20}=L_1+L_2$), yielding the result

$$\begin{bmatrix} (p\dot{0}) & (\dot{0}q) & (\lambda\dot{0}\mu)_S \\ l_1 & l_2 & (l_{10}l_{20})[L_1L_2]\end{bmatrix}$$

$$=(-1)^{(l_1+l_2-L_1-L_2)/2-\psi}2^{(\lambda+q+l_1+l_{20}+n-3)/2-l_2}$$

$$\times\frac{[(2l_1+n-2)(2l_2+n-2)]^{1/2}W'_n(p,l_1)W'_n(\lambda,l_{10})W'_n(\mu,l_{20})}{[\frac{1}{2}(\lambda+\mu+L_1+L_2)+n-3]!\,W_n(q,l_2)\nabla_{n[5,7]}(l_1l_2;L_1L_2)}$$

$$\times\left(\frac{(\lambda+\mu+n-1)!(p-\lambda)!(2L_1+n-2)!!(2L_2+n-4)!!}{(\lambda+q+n-1)!(2l_{10}+n-2)!!(2l_{20}+n-2)!!}\right.$$

$$\left.\times\frac{(L_1-l_{20})!(L_1+L_2+n-3)!}{(l_{20}-L_2)!}\right)^{1/2}$$

$$\times\sum_{x,y,z}\frac{(-1)^{x+y+z}[\frac{1}{2}(q-l_2)+x]!(2l_2+n-4-2x)!!}{x![\frac{1}{2}(p-\lambda-l_2+l_{20})+x]!z![\frac{1}{2}(l_1+l_2-L_1-L_2)-x-z]!}$$

$$\times\frac{(l_1-l_2+L_1+L_2+n-4+2x)!!}{(p+l_1+n-2-2z)!!}$$

$$\times\frac{[\frac{1}{2}(p-\lambda+l_1-l_{10})-z]![\frac{1}{2}(p-L_1-L_2+l_2)-x-z]!}{[\frac{1}{2}(p-\lambda+l_2-l_{20})-x-z]!y!(L_1-l_{20}-y)!}$$

$$\times\frac{[\frac{1}{2}(\lambda+\mu+L_1+L_2)+n-3+z]!}{[\frac{1}{2}(L_1-L_2-l_1+l_2)-y]!}$$

$$\times\frac{(L_1-L_2-y)!(l_1+l_2-L_1+L_2+n-4+2y)!!}{(l_1+l_2-L_1+L_2-n-4-2x+2y)!!}$$

$$\times\frac{(\lambda+q+l_{10}-l_1+L_1-L_2+n-2-2y)!!}{(\lambda+\mu+2L_1-l_1-l_2+n-2+2x-2y+2z)!!}. \tag{3.4}$$

Ališauskas (1983a) also obtained another expression for this isofactor, with a different kind of summation intervals. For special values of the parameters $p=\lambda$, $q=\mu$ the following expression in terms of SU_2 Clebsch–Gordan coefficients (for n even) or their analytical continuation (for n odd) is more convenient:

$$\begin{bmatrix} (\lambda\dot{0}) & (\dot{0}\mu) & (\lambda\dot{0}\mu)_S \\ l_1 & l_2 & (l_{10}l_{20})[L_1L_2] \end{bmatrix}$$
$$= (-1)^{(l_1+l_2-L_1-L_2)/2-\psi}2^{(L_1+L_2+n-4)/2}$$
$$\times\left(\frac{(2l_1+n-2)(2l_2+n-2)(L_1+L_2+n-3)!(2L_1+n-2)!!}{(L_1-L_2+1)(2l_{10}+n-2)!!(2l_{20}+n-2)!!}\right.$$
$$\left.\times\frac{(2L_2+n-4)!!A_7(l_1+l_{10}+n-4)!!(l_2+l_{20}+n-4)!!}{A_0A_1A_2(l_1-l_{10})!!(l_2-l_{20})!!}\right)^{1/2}$$
$$\times W_n(\lambda, l_{10})W_n(\mu, l_{20})/W_n(\lambda, l_1)W_n(\mu, l_2)$$
$$\times\begin{bmatrix} \frac{1}{2}l_1+\frac{1}{4}n-1 & \frac{1}{2}l_2+\frac{1}{4}n-1 & \frac{1}{2}(L_1-L_2) \\ \frac{1}{2}l_{10}+\frac{1}{4}n-1 & -\frac{1}{2}l_{20}-\frac{1}{4}n+1 & \frac{1}{2}(l_{10}-l_{20}) \end{bmatrix}. \tag{3.5}$$

Both quantities (3.4) and (3.5) gain no phase factor if permutation of the parameters

$$p\rightleftarrows q, \qquad \lambda\rightleftarrows\mu, \qquad l_1\rightleftarrows l_2, \qquad l_{10}\rightleftarrows l_{20} \tag{3.8}$$

is performed.

By means of the projection operators of the subgroup $SU_2\times SU_2$, the overlaps of the projected Draayer states of $SU_4\supset SU_2\times SU_2$ were expressed (Norvaišas 1981) as fourfold sums and transformed to a more convenient form (Ališauskas 1983a). The corresponding overlaps of stretched states of $SU_n\supset SO_n$ take the form

$$\left\langle \begin{matrix} (\lambda\dot{0}\mu)_S \\ (l_{10}l_{20})[L_1L_2] \end{matrix} \middle| \begin{matrix} (\lambda\dot{0}\mu)_S \\ (l'_{10}l'_{20})[L_1L_2] \end{matrix} \right\rangle$$
$$= 2^{[(L_1+L_2-\lambda-\mu)+3(l_{20}-l'_{20})]/2}(L_1+L_2+n-3)$$
$$\times\frac{(2L_1+n-2)!!(2L_2+n-4)!!(\lambda+l'_{10}+n-2)(\mu+l_{20}+n-2)}{[(2l_{10}+n-2)!!(2l_{20}+n-2)!!(2l'_{10}+n-2)!!(2l'_{20}+n-2)!!]^{1/2}}$$
$$\times\left(\frac{(l_{20}-L_2)!(L_1-l'_{20})!}{(l'_{20}-L_2)!(L_1-l_{20})!}\right)^{1/2}\frac{W'_n(\lambda, l_{10})W'_n(\mu, l'_{20})}{W'_n(\mu, l_{20})W'_n(\lambda, l'_{10})}$$
$$\times\sum_{x,y,z}\frac{(-1)^{x+y+z}[\frac{1}{2}(l_{20}-l'_{20})+x]!(\lambda+\mu+l_{10}-l_{20}+n-4-2x)!!}{x!(\lambda+\mu-L_1+L_2+n-4-2x)!!(\lambda+\mu+L_1-L_2+n-2-2x)!!}$$
$$\times\frac{(\lambda+\mu-l'_{10}+l'_{20}+n-4-2x)!!(2\lambda+n-4-2y)!!(2\mu+n-4-2z)!!}{y![\frac{1}{2}(\lambda-l'_{10})-y]!(\lambda+l_{10}+n-4-2y)!!z![\frac{1}{2}(\mu-l_{20})-z]!}$$
$$\times\frac{(y+z)![\frac{1}{2}(\lambda+\mu+l_{10}+l'_{20})+n-4-y-z]!}{(\mu+l'_{20}+n-4-2z)!!(y+z-x)!}$$
$$\times\{[\frac{1}{2}(\lambda+\mu+L_1+L_2)+n-3+x-y-z]!\}^{-1} \qquad (l_{20}\geqslant l'_{20}). \tag{3.7}$$

4. Intermezzo. New expansion of projected basis of five-dimensional quasispin

Smirnov and Tolstoy (1973) proposed a construction of the complete basis for the reduction $Sp_4\supset U_2$ ($SO_5\supset SO_3\dot{+}SO_2$), different from those by Ahmed and Sharp (1970).

The expansion coefficients of these basis states in terms of the canonical basis (corresponding to the reduction $Sp_4 \supset SU_2 \times SU_2$) were simplified by Ališauskas (1983b). The complementary group technique allowed us (Ališauskas 1983b) to find the corresponding expansion of the dual basis states as well. The substitutions (Ališauskas and Norvaišas 1980, see Ališauskas 1983c)

$$p \to -2M-2, \qquad q \to -2N-2, \qquad \lambda \to -V-T-2, \qquad \mu \to V-T-2,$$
$$S \to -K-\Lambda-2, \qquad T \to K-\Lambda, \qquad j_{10} \to -K+\alpha-1, \qquad j_{20} \to -\Lambda-\alpha-1 \tag{4.1}$$

into the expression for the $SU_4 \supset SU_2 \times SU_2$ isofactor, represented by (2.8), allowed one to find the following expansion coefficient:

$$\left\langle \begin{matrix} \langle K\Lambda\rangle \\ IMJN \end{matrix} \middle| \begin{matrix} \langle K\Lambda\rangle \\ \alpha;\, VTM_T \end{matrix} \right\rangle = \delta_{V,\,M-N}\,\delta_{M_T,\,M+N}\,\frac{(2T+1)E(K\Lambda\alpha VT)}{(K-\Lambda+T)!}$$
$$\times\left(\frac{(2T)!(2I+1)(2J+1)(T+M_T)!(I-M)!(J-N)!(2K+1)!}{(T-M_T)!(I+M)!(J+N)!(K+\Lambda-I-J)!(K+\Lambda+I-J+1)!}\right.$$
$$\left.\times\frac{(2\Lambda)!(2K+2\Lambda+2)!(K-\Lambda+I-J)!(K-\Lambda-I+J)!(I+J-K+\Lambda)!}{(K+\Lambda-I+J+1)!(K+\Lambda+I+J+2)!(K-\Lambda+I+J+1)!}\right)^{1/2}$$
$$\times\sum_{x,\,y,\,z,\,u}\frac{(-1)^{K+\Lambda-I-J+x+u}(K-\Lambda+T+z)!(T-M_T+x)!}{z!(x-y-z)!(y-z)!(2\alpha-x+y+z)!(2K-2\Lambda-2\alpha-y+z)!}$$
$$\times\frac{(2\alpha-x+2y)!(2K-2\Lambda-2\alpha+x-2y)!}{u!(I+J-K+\Lambda-u)![\frac{1}{2}(T-M_T)-K+\alpha+I+y-u]!(2T+x+1)!}$$
$$\times\frac{[K-\alpha+I-\frac{1}{2}(T-M_T)-y]![\Lambda+\alpha+J-\frac{1}{2}(T-M_T)-x+y]!}{[\frac{1}{2}(T-M_T)-\Lambda-\alpha+J+x-y-u]![K+\alpha-\frac{1}{2}(T-M_T)-I-x+y+u]!}$$
$$\times\{[2K-\Lambda-\alpha-\tfrac{1}{2}(T-M_T)-J-y+u]![K-\alpha-\tfrac{1}{2}(T+V)-y]!$$
$$\times[\Lambda+\alpha-\tfrac{1}{2}(T-V)-x+y]!\}^{-1} \tag{4.2}$$

where

$$E(K\Lambda\alpha VT) = \Big([K-\alpha-\tfrac{1}{2}(T+V)]![K-\alpha+\tfrac{1}{2}(T+V)![\Lambda+\alpha-\tfrac{1}{2}(T-V)]!$$
$$\times[\Lambda+\alpha+\tfrac{1}{2}(T-V)]!\frac{(2K-2\Lambda-2\alpha)!(2\alpha)!}{(2K-2\alpha)!(2\Lambda+2\alpha)!}\Big)^{1/2}. \tag{4.3}$$

The projected (renormalised) states of $Sp_4 \supset U_2$ here are obtained by the action of the projection operator of the subgroup SU_2 into special canonical basis states (as defined by Ališauskas and Jucys 1969, 1971)

$$\left| \begin{matrix} \langle K\Lambda\rangle \\ \alpha;\, VTM_T \end{matrix} \right\rangle = P^T_{M_T T}\left| \begin{matrix} \langle K\Lambda\rangle \\ K-\alpha, \frac{1}{2}(T+V), \Lambda+\alpha, \frac{1}{2}(T-V) \end{matrix} \right\rangle. \tag{4.4}$$

The relations between our parameters and those of Smirnov and Tolstoy (1973) are as follows:

$$f_1 = K+\Lambda, \qquad f_2 = K-\Lambda, \qquad p_1 = K-\alpha+\tfrac{1}{2}(T-V), \qquad p_2 = 2\alpha. \tag{4.5}$$

The overlaps of the projected states may be expressed as particular cases of (4.2) with $I+J=K+\Lambda$, $\alpha'=K-I=J-\Lambda$, $M+N=M_T=T$. It is not difficult to obtain the expansion coefficients for symmetric irreps $\langle K0\rangle$ and $\langle \Lambda\Lambda\rangle$ from (4.2). They take the forms of ${}_3F_2(1)$ and Saalschutzian ${}_4F_3(1)$ series (see Slater 1966), respectively (cf Ahmed and Sharp 1970, Ališauskas 1983b).

The linearly independent states are those with parameters†

$$\alpha \leqslant A = \min[T-\tfrac{1}{2}\delta_1, \Lambda+\tfrac{1}{2}(T-V)], \tag{4.6}$$

$$\delta_1 = 0 \text{ or } 1, \qquad 2K-T+V-\delta_1 \text{ being even.}$$

The expansion coefficient of linearly dependent states takes the following form (cf Smirnov and Tolstoy 1973, Ališauskas 1983b):

$$Q^{(K\Lambda)VT}_{\alpha'\alpha} = \frac{(-1)^{\alpha'-A}}{(\alpha'-\alpha)(A-\alpha)!(\alpha'-A-1)!}\frac{(K-\Lambda-T+A-\alpha)!E(K\Lambda\alpha'VT)}{(K-\Lambda-T+A-\alpha')!E(K\Lambda\alpha VT)}. \tag{4.7}$$

In order to derive (4.7) both the main and superfluous states should be expanded in terms of states of the auxiliary basis related to the symmetric basis of Ahmed and Sharp (1970) and the matrix of the inverse transformation should be found. The expansion coefficients (4.7) will be needed later in § 6 and appendix 1.

5. Projected states of two-rowed irreps and isofactors for coupling $(p_1\dot{0})\times(p_2\dot{0})$ to $(\lambda\nu\dot{0})$

The states of the two-parametric covariant representations of $SU_n \supset SO_n$ are also the states for the chain of the complementary groups $Sp(4, R) \supset U_2$. Therefore the substitutions (Ališauskas and Norvaišas 1980, see Ališauskas 1983c)

$$p_1 \to p_1+\tfrac{1}{2}n-2, \qquad p_2 \to p_2+\tfrac{1}{2}n-2, \qquad \lambda\to\lambda, \qquad \nu\to\nu+\tfrac{1}{2}n-2, \tag{5.1}$$

together with (2.7) and the corresponding transformation of the multiplicity labels, permit us to find the isofactors of $SU_n \supset SO_n$ for coupling $(p_1\dot{0})\times(p_2\dot{0})$ to $(\lambda\nu\dot{0})$ if the corresponding expressions for isofactors of $SU_4 \supset SU_2\times SU_2$ are available.

The expression for isofactors needed for coupling to the projected basis states of $SU_4 \supset SU_2\times SU_2$ was proposed by Ališauskas and Norvaišas (1980) and simpiified by Ališauskas (1982). It can be written for $SU_n \supset SO_n$ in the following form ($p_1+p_2=\lambda+2\nu$):

$$\begin{bmatrix} (p_1\dot{0}) & (p_2\dot{0}) & (\lambda\nu\dot{0})_E \\ l_1 & l_2 & k[L_1L_2] \end{bmatrix}$$
$$= G(n)(-1)^{(l_1+l_2-L_1-L_2)/2+\varphi}2^{-(l_1+3l_2-1)/2}$$
$$\times[(2l_1+n-2)(2l_2+n-2)(L_1+L_2+n-3)(L_1-L_2+1)$$
$$\times(2\nu+n-4)!!(2\lambda+2\nu+n-2)!!]^{1/2}$$
$$\times\left(\frac{[\frac{1}{2}(L_1-L_2)-k]!(L_1+L_2+n-4+2k)!!}{[\frac{1}{2}(L_1-L_2)+k]!(L_1+L_2+n-4-2k)!!}\right)^{1/2}$$

† Unfortunately, in the corresponding equation (2.7) (Ališauskas 1983b), δ_1 is omitted.

$$\times \frac{\nabla_{n[1,3,4,6]}(l_1 l_2; L_1 L_2)}{W_n(p_1, l_1) W_n(p_2, l_2)}$$

$$\times \sum_{x,y} \frac{(-1)^x 2^x (2l_2+n-4-2x)!!(l_1+l_2+n-4-2k-2x)!!}{x![\frac{1}{2}(l_1+l_2-L_1-L_2)-x]![\frac{1}{2}(l_1+l_2+L_1+L_2)+n-3-x]!}$$

$$\times \frac{[\frac{1}{2}(l_1-l_2+L_1-L_2)+y]!(2l_2+n-4-2y)!!}{y![\frac{1}{2}(L_1-L_2-l_1+l_2)-y]![\frac{1}{2}(l_1-l_2)-k+y]!(2l_2+n-4-2x-2y)!!}$$

$$\times \left(\frac{[\frac{1}{2}(p_1-l_2)+k+x]![\frac{1}{2}(p_2-l_2)+x]!}{(\lambda+\nu-l_2+x+y+1)!} \right.$$

$$\left. \times \frac{[\frac{1}{2}(p_1-l_2)-k+y]![\frac{1}{2}(p_2-l_2)+y]!}{(\nu-l_2+x+y)!} \right)^{1/2}$$

$$\times \begin{bmatrix} \frac{1}{2}(p_1-l_2+x+y) & \frac{1}{2}(p_2-l_2+x+y) & \frac{1}{2}\lambda \\ k+\frac{1}{2}(x-y) & \frac{1}{2}(y-x) & k \end{bmatrix}. \tag{5.2}$$

On the RHS the Clebsch–Gordan coefficients of SU_2 appeared. $G(n)$ is some normalisation constant and may be omitted. The parameters of isofactors considered in §§ 5–7 may be permutted ($p_1 \rightleftarrows p_2$, $l_1 \rightleftarrows l_2$), changing the isofactor only by a phase

$$(-1)^{\nu-L_2}. \tag{5.3}$$

The multiplicity label k is equal in the $SU_4 \supset SU_2 \times SU_2$ case to the Draayer (1970) labels $K_S = K_T$. In the $SU_3 \supset SO_3$ case, the states obtained correspond to Elliott's states (E^- as defined by Ališauskas (1978)) $K = 2k$, up to an elementary renormalisation factor (for details and additional information see Ališauskas (1982)).

The meaning of the multiplicity label k is not yet obvious for $n \geq 5$, but the bilinear combinations of isofactors may be expanded in terms of quantities (5.2) with non-negative $k \geq \frac{1}{2}(L_1 - \nu)$ (cf Ališauskas and Norvaišas 1980). Then the states with $k \geq \frac{1}{2}(L_1 - \nu)$, $k \geq 0$ (k may be 0 only for both $\nu - L_2$ and λ even) form a complete system (cf Draayer 1970). The corresponding overlaps, found by Ališauskas (1982), are given by

$$\left\langle \begin{matrix} (\lambda\nu\dot{0})_E \\ k[L_1 L_2] \end{matrix} \middle| \begin{matrix} (\lambda\nu\dot{0})_E \\ k'[L_1 L_2] \end{matrix} \right\rangle$$

$$= G^2(n) \frac{(L_1+L_2+n-3)(L_1-L_2+1)}{2^{(\lambda+L_1+L_2)/2+2\nu+\Delta_0+n-4}}$$

$$\times \frac{(2\lambda+2\nu+n-2)!!(2\nu+n-4)!!(2k)!(2k')^{\Delta_0}}{(\lambda+\nu-L_2+1)!(\nu+L_2+\Delta_0+n-5)!!}$$

$$\times \left(\frac{(\frac{1}{2}\lambda+k')!(\frac{1}{2}\lambda-k')![\frac{1}{2}(L_1-L_2)+k']![\frac{1}{2}(L_1-L_1)-k']!}{(\frac{1}{2}\lambda+k)!(\frac{1}{2}\lambda-k)![\frac{1}{2}(L_1-L_2)+k]![\frac{1}{2}(L_1-L_2)-k]!} \right.$$

$$\times \frac{(L_1+L_2-2k+n-4)!!(L_1+L_2+2k'+n-4)!!}{(L_1+L_2+2k+n-4)!!}$$

$$\left. \times \frac{(L_1+L_2-2k'+n-4)!!}{1} \right)^{1/2}$$

$$\times \sum_{l_2, l_0, x} \frac{(-1)^{(l_0+l_2-\nu-L_2-\Delta_0)/2}[\frac{1}{2}(l_2-L_2-\Delta_0)]![\frac{1}{2}(\lambda+\nu-l_2)-k]!}{[\frac{1}{2}(\nu-l_2)]![\frac{1}{2}(\lambda+\nu-L_1+L_2-l_2)]!(L_1+l_2+n-3)!(2x+\Delta_0)!}$$
$$\times \frac{[\frac{1}{2}(\lambda+\nu-L_2-\Delta_0)+k-x]!(L_1+l_2+2k-\Delta_0+n-4-2x)!!}{(2k-\Delta_0-2x)![\frac{1}{2}(l_2-L_2-\Delta_0)-x]!(\lambda+\nu+L_1-\Delta_0+n-2-2x)!!}$$
$$\times \frac{2^{l_0}(L_1-l_0)!(l_2+l_0+n-5)!!}{[\frac{1}{2}(\nu-l_0)]![\frac{1}{2}(L_1+\Delta_0-l_0)-k']![\frac{1}{2}(L_1+\Delta_0-l_0)+k']!}$$
$$\times \{[\tfrac{1}{2}(l_0+l_2-\nu-L_2-\Delta_0)]!(L_2-\Delta_0+l_0+n-4)!!\}^{-1} \qquad (k \leqslant k'). \tag{5.4}$$

Here $\Delta_0 = 0$ or 1, so that $\lambda+\nu-L_2-\Delta_0$ is even. The indeterminacy of $(-2)!!/(-2)!!$ which appears for $n=3$, $\nu=0$ (in this case $L_2=\Delta_0=l_0=l_2=0$) should be replaced by 1.

6. Boundary value problems for coupling $(p_1\dot{0})\times(p_2\dot{0})$ to $(\lambda\nu\dot{0})$

Let us consider the quantities obtained from (3.4) by use of the special elements of the substitution groups (Ališauskas and Jucys 1967, 1974) of the parameters of irreps of both SU_n and SO_n:

$$\lambda \to -\lambda-\nu-n, \qquad \mu \to \nu, \qquad p \to -p_1-n, \qquad q \to p_2, \tag{6.1}$$

$$L_1 \to -L_1-n+2, \qquad L_2 \to L_2, \qquad l_1 \to -l_1-n+2, \qquad l_2 \to l_2, \qquad l_{20} \to \underline{l}_2. \tag{6.2}$$

These quantities are expansion coefficients of isofactors for coupling $(p_1\dot{0})\times(p_2\dot{0})$ to $(\lambda\nu\dot{0})$ in terms of their boundary values in the region

$$p_1 = \lambda+\nu, \qquad p_2 = \nu, \qquad l_1 - l_2 = L_1 - L_2. \tag{6.3}$$

The cardinality of this region is equal to or greater than the multiplicity of the corresponding irrep of the subgroup (cf Ališauskas 1978, 1983c). Such quantities are proposed (Ališauskas 1983c) to be called pseudoisofactors. The proper isofactors may be expanded in terms of pseudoisofactors if the corresponding values of isofactors in all above-mentioned boundary regions are known. The deduction discussed in appendix 1 leads to the following expression for the isofactors which couple to antistretched (A) basis states $(\underline{l}_1 = L_1 - L_2 + \underline{l}_2, p_1+p_2 = \lambda+2\nu)$:

$$\begin{bmatrix} (p_1\dot{0}) & (p_2\dot{0}) & (\lambda\nu\dot{0})_A \\ l_1 & l_2 & (\underline{l}_2)[L_1L_2] \end{bmatrix}$$
$$= (-1)^{\psi'+p_2-l_2} \frac{[(2l_1+n-2)(2l_2+n-2)]^{1/2}}{2^{(L_1-p_2+l_1+\underline{l}_2-\Delta_0+n-3)/2-L_2+l_2}}$$
$$\times \left(\frac{(p_1-\nu)!(p_2-\nu)!(2L_2+n-4)!!(2\underline{l}_1+n-4)!!(\underline{l}_2-L_2-\Delta_0)!!}{\lambda!(2L_1+n-4)!!(L_1-L_2)!(2\underline{l}_2+n-2)!!(\underline{l}_2-L_2+\Delta_0-1)!!} \right.$$
$$\left. \times \frac{(L_1+\underline{l}_2-\delta_0+n-3)!!}{(L_1+\underline{l}_2+\delta_0+n-4)!!} \right)^{1/2} \frac{\nabla_{n[1,3]}(l_1l_2;L_1L_2)\,W'_n(\nu,\underline{l}_2)}{W'_n(p_1,l_1)\,W_n(p_2,l_2)\,W_n(\lambda+\nu,\underline{l}_1)}$$

$$\times \sum_{l_0} \frac{[\frac{1}{2}(\lambda + L_1 - L_2) + 1]! [\frac{1}{2}(\lambda + \nu - L_1 + L_2 - l_0)]!}{[\frac{1}{2}(l_0 - L_2 - \Delta_0)]! (L_1 + l_0 + n - 3 - \delta_0)!!} R_{l_2 l_0}$$

$$\times \sum_{x, y, z} \frac{(-1)^{y+z} [\frac{1}{2}(p_2 - l_2) + x]! (2l_2 + n - 4 - 2x)!!}{x! y! z! [\frac{1}{2}(p_2 - \nu - l_2 + l_0) + x]! [\frac{1}{2}(L_1 - L_2 - l_1 + l_2) - x - z]!}$$

$$\times \frac{[\frac{1}{2}(p_2 - \nu + L_1 - L_2 - l_1 + l_0) - z]!}{[\frac{1}{2}(p_2 - \nu + l_2 - l_0) - x - z]! [\frac{1}{2}(p_1 - L_1 + L_2 - l_2) + x + z]!}$$

$$\times \frac{(p_1 + l_1 + n - 2 + 2z)!! (L_1 + l_0 + n - 3 + y)!}{[\frac{1}{2}(\lambda + L_1 - L_2) - z + 1]! (l_1 + l_2 + L_1 - L_2 + n - 2 - 2x)!!}$$

$$\times \frac{(L_1 + L_2 - l_1 + l_2 + n - 4 + 2y)!!}{[\frac{1}{2}(l_1 + l_2 - L_1 - L_2) - y]! (L_1 + L_2 + n - 3 + y)!}$$

$$\times \frac{(\lambda + 2L_1 - l_1 + l_2 + n - 2 - 2x + 2y - 2z)!!}{(L_1 + L_2 - l_1 + l_2 + n - 4 - 2x + 2y)!! (p_1 - \nu + 2L_1 - l_1 + l_0 + n - 2 + 2y)!!}. \qquad (6.4)$$

Here

$$R_{l_2 l_0} = \delta_{l_2 l_0} + \frac{2(-1)^{(\nu + \delta_0 + \Delta_0 - L_1 + L_2 - l_0)/2}}{(l_2 - l_0)[\frac{1}{2}(L_1 - L_2 - \nu - \delta_0 + l_2)]! [\frac{1}{2}(\nu + \Delta_0 + \delta_0 - L_1 + L_2 - l_0) - 1]!}. \qquad (6.5)$$

For special values of the parameters $p_1 = \lambda + \nu$, $p_2 = \nu$ the following expression obtained by means of analytical continuation of (3.5) may be more useful:

$$\begin{bmatrix} (\lambda + \nu\dot{0}) & (\nu\dot{0}) & (\lambda\nu\dot{0})_A \\ l_1 & l_2 & (\underline{l}_2)[L_1 L_2] \end{bmatrix}$$

$$= (-1)^{(l_1 - l_2 - L_1 + L_2)/2 - \psi'} 2^{L_2 + (\nu - \underline{l}_2 + n - 5)/2}$$

$$\times \left[\frac{(2l_1 + n - 2)(2l_2 + n - 2)(2L_2 + n - 4)!!(2\underline{l}_1 + n - 4)!!(\underline{l}_2 - L_2 - \Delta_0)!!}{(L_1 - L_2)!(2L_1 + n - 4)!!(2\underline{l}_2 + n - 2)!!(\underline{l}_2 - L_2 + \Delta_0 - 1)!!} \right.$$

$$\times \left. \frac{(L_1 + \underline{l}_2 - \delta_0 + n - 3)!!}{(L_1 + \underline{l}_2 + \delta_0 + n - 4)!!} \right]^{1/2} \frac{W'_n(\nu, \underline{l}_2) W_n(\lambda + \nu, l_1)}{W_n(\lambda + \nu, \underline{l}_1) W_n(\nu, l_2)}$$

$$\times \nabla_{n[0,3]}(l_1 l_2; L_1 L_2) \sum_{l_0} \frac{[\frac{1}{2}(\nu - l_0)]! (l_0 - L_2 + \Delta_0 - 1)!! (l_2 + l_0 + n - 4)!!}{(L_1 + l_0 - \delta + n - 3)!! (l_1 + L_1 - L_2 + l_0 + n - 2)!!}$$

$$\times R_{l_2 l_0} \sum_z \frac{2^{-z} [L_1 + \frac{1}{2}(l_2 + l_0) + n - 3 - z]!}{[\frac{1}{2}(L_1 - L_2 - l_1 + l_0) - z]! [\frac{1}{2}(l_2 - l_0) - z]!}$$

$$\times \{[\tfrac{1}{2}(l_1 - L_1 - L_2 + l_0) + z]! (L_1 + L_2 - l_1 + l_2 + n - 4 - 2z)!!\}^{-1}. \qquad (6.6)$$

The antistretched states introduced by Moshinsky *et al* (1975; see Ališauskas 1978, Ališauskas and Norvaišas 1980) are defined by the boundary condition: the special isofactor (6.6) taking values $\delta_{l_2 \underline{l}_2}$ if $l_1 = L_1 - L_2 + l_2$ and

$$l_2 \geqslant \nu - L_1 + L_2 + \delta_0 + \Delta_0, \qquad \underline{l}_2 \geqslant \nu - L_1 + L_2 + \delta_0 + \Delta_0. \qquad (6.7)$$

Here $\delta_0 = 0$ or 1, $\Delta_0 = 0$ or 1, so that $\nu - L_2 - \delta_0$ and $\lambda + \nu - L_2 - \Delta_0$ are even.

Another solution of the boundary value problem for isofactors under consideration may be found by substituting (6.1) into (3.4) without the substitution (6.2). The pseudoisofactors obtained in such a way are expansion coefficients of isofactors for the coupling $(p_1\dot{0})\times p_2\dot{0})$ to $(\lambda\nu\dot{0})$ in terms of their boundary values in the region

$$p_1=\lambda+\nu, \qquad p_2=\nu, \qquad l_1+l_2=L_1+L_2. \tag{6.8}$$

Thus (see appendix 1) the general expression for the proper isofactors which couple to the quasistretched (Q) basis states was obtained $(\bar{l}_1+\bar{l}_2=L_1+L_2)$

$$\begin{aligned}
&\begin{bmatrix} (p_1\dot{0}) & (p_2\dot{0}) & (\lambda\nu\dot{0})_Q \\ l_1 & l_2 & (\bar{l}_2)[L_1L_2] \end{bmatrix} \\
&\quad=(-1)^{p_2-\nu+(l_1+l_2-L_1-L_2)/2-\psi}2^{(p_1+p_2+L_1+L_2+n-3)/2-l_2} \\
&\quad\times\left(\frac{(2l_1+n-2)(2l_2+n-2)(p_1-\nu)!(p_2-\nu)!(L_1+L_2+n-3)!(2L_1+n-2)!!}{\lambda!(2\bar{l}_1+n-2)!!(2\bar{l}_2+n-2)!!(\bar{l}_2-L_2)!}\right. \\
&\quad\times\left.\frac{(2L_2+n-4)!!}{(L_1-\bar{l}_2)!}\right)^{1/2}\frac{W'_n(p_1,l_1)\,W'_n(\lambda+\nu,\bar{l}_1)\,W'_n(\nu,\bar{l}_2)}{\nabla_{n[5,7]}(l_1l_2,L_1L_2)\,W_n(p,l_2)} \\
&\quad\times\sum_{\bar{l}_0}\frac{[\frac{1}{2}(\bar{l}_2-L_2-\Delta_0)]!(L_1-\bar{l}_0)!}{[\frac{1}{2}(\bar{l}_0-L_2-\Delta_0)]!}R'_{\bar{l}_2\bar{l}_0}\sum_{x,y,z}\frac{(-1)^{x+y+z}}{x!y!z!} \\
&\quad\times\frac{[\frac{1}{2}(p_2-l_2)+x]!(2l_2+n-4-2x)!!(L_1+L_2+l_1-l_2+n-4+2x)!!}{[\frac{1}{2}(l_1+l_2-L_1-L_2)-x-z]![\frac{1}{2}(p_2-\nu+l_2-\bar{l}_0)-x-z]![\frac{1}{2}(p_2-\nu-l_2+\bar{l}_0)+x]!} \\
&\quad\times\left\{\frac{(\lambda-L_1-L_2-n+4)!!}{(\lambda-L_1-L_2-n+4-2z)!!}\right\}\frac{[\frac{1}{2}(p_1-l_1)+z]!(L_1-L_2-y)!}{(p_1+L_1+L_2-l_2+n-2+2x+2z)!!} \\
&\quad\times\frac{[\frac{1}{2}(p-\nu+l_1-L_1-L_2+\bar{l}_0)-z]![\frac{1}{2}(\lambda+l_1+l_2)-L_1-x+y-z]!}{(L_1-\bar{l}_0-y)![\frac{1}{2}(L_1-L_2-l_1+l_2)-y]![\frac{1}{2}(p_1-\nu+l_1+\bar{l}_0)-L_1+y]!} \\
&\quad\times\frac{(l_1+l_2-L_1+L_2+n-4+2y)!!}{(l_1+l_2-L_1+L_2+n-4-2x+2y)!!}
\end{aligned}\tag{6.9}$$

when $\lambda-L_1-L_2-n+4\geqslant 0$. Otherwise the quantity in braces in (6.9) should be replaced by

$$(-1)^z(L_1+L_2-\lambda+n-6+2z)!!/(L_1+L_2-\lambda+n-6)!!. \tag{6.10}$$

Here

$$R'_{\bar{l}_2\bar{l}_0}=\delta_{\bar{l}_2\bar{l}_0}+\frac{2(-1)^{(\lambda-L_1+\bar{l}_2-\Delta_0)/2}}{(\bar{l}_2-\bar{l}_0)[\frac{1}{2}(\lambda-L_1+\bar{l}_2-\Delta_0)]![\frac{1}{2}(L_1-\lambda-\bar{l}_0+\Delta_0)-1]!} \tag{6.11}$$

The expression

$$\begin{aligned}
&\begin{bmatrix} (\lambda+\nu,\dot{0}) & (\nu\dot{0}) & (\lambda\nu\dot{0})_Q \\ l_1 & l_2 & (\bar{l}_2)[L_1L_2] \end{bmatrix} \\
&\quad=(-1)^{(l_1+l_2-L_1-L_2)/2-\psi}2^{(L_1+L_2+n-4)/2} \\
&\quad\times\left(\frac{(2l_1+n-2)(2l_2+n-2)(L_1+L_2+n-3)!(2L_1+n-2)!!A_7}{(L_1-L_2+1)(2\bar{l}_1+n-2)!!(2\bar{l}_2+n-2)!!(\bar{l}_2-L_2)!(L_1-\bar{l}_2)!}\right. \\
&\quad\times\left.\frac{(2L_2+n-4)!!}{A_0A_1A_2}\right)^{1/2}\frac{W_n(\lambda+\nu,l_1)\,W'_n(\lambda+\nu,\bar{l}_1)\,W'_n(\nu,\bar{l}_2)}{W_n(\nu,l_2)} \\
&\quad\times\sum_{\bar{l}_0}\frac{[\frac{1}{2}(\bar{l}_2-L_2-\Delta_0)]!(\nu-\bar{l}_0)!!}{[\frac{1}{2}(l_0-L_2-\Delta_0)]!(\lambda+\nu+L_1+L_2-\bar{l}_0+n-2)!!}R'_{\bar{l}_2\bar{l}_0}
\end{aligned}$$

$$\times\left(\frac{(l_1+L_1+L_2-\bar{l}_0+n-4)!!(l_2+\bar{l}_0+n-4)!!(\bar{l}_0-L_2)!(L_1-\bar{l}_0)!}{(l_1-L_1-L_2+\bar{l}_0)!!(l_2-\bar{l}_0)!!}\right)^{1/2}$$

$$\times\begin{bmatrix}\frac{1}{2}l_1+\frac{1}{4}n-1 & \frac{1}{2}l_2+\frac{1}{4}n-1 & \frac{1}{2}(L_1-L_2)\\ \frac{1}{2}(L_1+L_2-\bar{l}_0)+\frac{1}{4}n-1 & -\frac{1}{2}\bar{l}_0-\frac{1}{4}n+1 & \frac{1}{2}(L_1+L_2)-\bar{l}_0\end{bmatrix} \qquad (6.12)$$

appears for special values of parameters.

The quasistretched states are defined by the boundary condition: the special isofactor (6.12) taking the values $\delta_{l_2\bar{l}_2}$ if $l_1=L_1+L_2-l_2$ and

$$l_2\geqslant L_1-\lambda+\Delta_0, \qquad \bar{l}_2\geqslant L_1-\lambda+\Delta_0. \qquad (6.13)$$

The pseudoisofactors appear if $R_{\underline{l}_2 l_0}$ or $R'_{\bar{l}_2\bar{l}_0}$ in (6.4), (6.6), (6.9) and (6.12) are replaced by $\delta_{\underline{l}_2 l_0}$ or $\delta_{\bar{l}_2\bar{l}_0}$. The pseudoisofactors coincide with proper isofactors if $\nu\leqslant L_1-\delta_0$ or $L_1-L_2\leqslant\lambda$ in the first or second case, respectively.

7. Bilinear combinations of isofactors for coupling $(p_1\dot{0})\times(p_2\dot{0})$ to $(\lambda\nu\dot{0})$

Procedures analogous to analytical continuation which was used to obtain (6.4), applied to (2.8) allowed us to obtain the following expression for bilinear combinations of isofactors ($\underline{l}_1=L_1-L_2+\underline{l}_2$; $p_1+p_2=\lambda+2\nu$):

$$\begin{bmatrix}(p_1\dot{0}) & (p_2\dot{0}) & (\lambda\nu\dot{0})_{\bar{A}}\\ l_1 & l_2 & (\underline{l}_2)[L_1L_2]\end{bmatrix}$$

$$=\sum_{\omega}\begin{bmatrix}(\lambda+\nu\dot{0}) & (\nu\dot{0}) & (\lambda\nu\dot{0})\\ \underline{l}_1 & \underline{l}_2 & \omega[L_1L_2]\end{bmatrix}\begin{bmatrix}(p_1\dot{0}) & (p_2\dot{0}) & (\lambda\nu\dot{0})\\ l_1 & l_2 & \omega[L_1L_2]\end{bmatrix}$$

$$=\left(\frac{\lambda!(p_1-\nu)!(2L_1+n-4)!!(2\underline{l}_2+n-2)!!(L_1-L_2)!(\underline{l}_2-L_2)!}{(p_2-\nu)!(2L_2+n-4)!!(2\underline{l}_1+n-4)!!(L_1+\underline{l}_2+n-3)!}\right)^{1/2}$$

$$\times[(2l_1+n-2)(2l_2+n-2)]^{1/2}\nabla_{n[0,4,5,6]}(l_1l_2;L_1L_2)(\lambda+1)2^{(\lambda+L_1+L_2+n-3)/2}$$

$$\times\frac{W'_n(\lambda+\nu,\underline{l}_1)W_n(p_2,l_2)}{W'_n(\nu,\underline{l}_2)W_n(p_1,l_1)}$$

$$\times\sum_{x,y,z,u}\frac{(-1)^{\psi'+x}2^{x-z-u}(L_1+\underline{l}_2+n-3+x-y-z)!}{z!(x-y-z)!(y-z)!(\underline{l}_2-L_2-y+z)!}$$

$$\times\left\{\frac{(L_1+L_2-\lambda+n-4)!!}{(L_1+L_2-\lambda+n-4-2x)!!}\right\}$$

$$\times\frac{(p_2-\nu+x)![\frac{1}{2}(\lambda+\nu-\underline{l}_1)+y]!}{(\lambda+x+1)!(L_1+\underline{l}_2+n-3+x-2y)![\frac{1}{2}(\nu-L_2)-x+y]!}$$

$$\times\frac{(\underline{l}_2-L_2+x-2y)!(\underline{l}_1+l_1-p_2+\nu+n-4-2y)!!}{(p_2-\nu+l_2+\underline{l}_2+n-2+2x-2y)!!u![\frac{1}{2}(p_2-\nu-\underline{l}_1+l_1)+y-u]!}$$

$$\times\frac{[\frac{1}{2}(L_1+L_2+l_1+\underline{l}_2+p_2-\nu)+n-3+x-y-u]!!}{[\frac{1}{2}(p_2-\nu+\underline{l}_2-l_2)+x-y-u]![\frac{1}{2}(l_2+\underline{l}_2-p_2+\nu)-L_2-y+u]!}$$

$$\times[(l_1-l_2+L_1+L_2+n-4-2u)!!]^{-1} \qquad (7.1)$$

if $L_1+L_2-\lambda+n-4\geqslant 0$. Otherwise the quantity in braces should be replaced by

$$(-1)^z(\lambda-L_1-L_2-n+2+2z)!!/(\lambda-L_1-L_2-n+2)!!. \tag{7.2}$$

The system A of non-orthogonal isofactors with the multiplicity label $\underline{l}_2$ may be overcomplete. The values of the parameter $\underline{l}_2$ satisfying the condition (6.7) correspond to a complete system of linearly independent isofactors. This is evident from the expression similar to (3.2) for general bilinear combinations of isofactors which couple irreps $(p_1\dot{0})\times(p_2\dot{0})$ or $(p_1'\dot{0})\times(p_2'\dot{0})$ to $(\lambda\nu\dot{0})$ in terms of the biorthogonal system of isofactors represented by (7.1) and (6.4).

The sum should be taken over the summation parameter $\underline{l}_2$, satisfying the condition (6.7), but in the case where $\tilde{A}$ is overcomplete it is possible to use the pseudoisofactor in the RHS of the analogue of equation (3.2) and to take the sum over all values of the parameter $\underline{l}_2$, including these corresponding to the linearly dependent states of the system $\bar{A}$.

The methods discussed in appendix 2 allowed us to obtain the following expression for overlaps ($\underline{l}_1'=L_1-L_2+\underline{l}_2'$, $\underline{l}_2\leqslant\underline{l}_2'$):

$$\left\langle\begin{matrix}(\lambda\nu\dot{0})_{\bar{A}}\\(\underline{l}_2)[L_1L_2]\end{matrix}\middle|\begin{matrix}(\lambda\nu\dot{0})_{\bar{A}}\\(\underline{l}_2')[L_1L_2]\end{matrix}\right\rangle
=\frac{W_n'(\lambda+\nu,\underline{l}_1)\,W_n(\lambda+\nu,\underline{l}_1')}{W_n(\nu,\underline{l}_2)\,W_n'(\nu,\underline{l}_2')}$$
$$\times\frac{(\lambda+1)!(L_1-L_2)!(2L_1+n-4)!!(\nu+L_2+\Delta_0+n-3)!!}{(2L_2+n-4)!!(\lambda+\nu-L_2+1)!2^{L_1+\underline{l}_2-\nu-(L_2-\underline{l}_2'-\Delta_0)/2}}$$
$$\times\left(\frac{(\underline{l}_2'-L_2+\Delta_0-1)!!(L_1+\underline{l}_2'+n-3)!(2\underline{l}_2'+n-2)!!}{(2\underline{l}_1'+n-4)!!(\underline{l}_2'-L_2-\Delta_0)!!(2\underline{l}_1+n-4)!!}\right.$$
$$\left.\times\frac{(\underline{l}_2-L_2)!(2\underline{l}_2+n-2)!!}{(L_1+\underline{l}_2+n-3)!}\right)^{1/2}$$
$$\times\sum_{k,t,z}\frac{(-1)^{(\underline{l}_2-\underline{l}_2')/2+k-t}(2k)^{1-\Delta_0}(k+t-1)![\frac{1}{2}(\nu-L_1-\Delta_0)+t]!}{(\frac{1}{2}\lambda+k)!(\frac{1}{2}\lambda-k)![\frac{1}{2}(L_1-L_2)-k]![\frac{1}{2}(L_1-L_2+k)]!(k-t)!}$$
$$\times\frac{2^{2t}(2L_1+2L_2+n-4-2t)!![\frac{1}{2}(\lambda+\nu-\underline{l}_2)-k]!(2k)!}{(2t-\Delta_0)!(L_1+\underline{l}_2'+\Delta_0+n-3-2t)!!(2L_1+2L_2+n-4+2k)!!}$$
$$\times\frac{[\frac{1}{2}(\lambda+\nu-L_2-\Delta_0)+k-z]!(L_1+\underline{l}_2-\Delta_0+n-4+2k-2z)!!}{[\frac{1}{2}(\nu-\underline{l}_1')+t]!(2z+\Delta_0)!(2k-\Delta_0-2z)![\frac{1}{2}(\underline{l}_2-L_2-\Delta_0)-z]!}$$
$$\times[(\lambda+\nu+L_1-\Delta_0+n-2-2z)!!]^{-1}. \tag{7.3}$$

The sum both over t and over z is a Saalschutzian ${}_4F_3(1)$ series (see Slater 1966). The number of summands in every sum never exceeds the multiplicity of the irrep $[L_1L_2]$ of SO_n in the irrep $(\lambda\nu\dot{0})$ of SU_n as well as in the expression (5.4) for overlaps of the projected states. The indeterminate quantity $(2k)(k+t-1)!$ which appears in the case $\Delta_0=0$, $k=0$, $t=0$ should be taken as 1.

The above-presented expressions of isofactors for coupling $(p_1\dot{0})\times(p_2\dot{0})$ to $(\lambda\nu\dot{0})$ do not simplify considerably for low-dimensional irreps of the subgroup SO_n. In this case another expression obtained by the analytical continuation of (2.8) (by a procedure similar to that used for (6.9)) may be more useful ($\bar{l}_1+\bar{l}_2=L_1+L_2$):

$$\begin{bmatrix}(p_1\dot{0}) & (p_2\dot{0}) & (\lambda\nu\dot{0})_{\bar{Q}} \\ l_1 & l_2 & (\bar{l}_1\bar{l}_2)[L_1L_2]\end{bmatrix}$$
$$=\sum_{\omega}\begin{bmatrix}(\lambda+\nu\dot{0}) & (\nu\dot{0}) & (\lambda\nu\dot{0}) \\ \bar{l}_1 & \bar{l}_2 & \omega[L_1L_2]\end{bmatrix}\begin{bmatrix}(p_1\dot{0}) & (p_2\dot{0}) & (\lambda\nu\dot{0}) \\ l_1 & l_2 & \omega[L_1L_2]\end{bmatrix}$$
$$\times\frac{[(2l_1+n-2)(2l_2+n-2)]^{1/2}(\lambda+1)}{\nabla_{n[4,7]}[\frac{1}{2}(\lambda+L_1-L_2)]!\,W_n(p_1,l_1)\,W'_n(\lambda+\nu,\bar{l}_1)}$$
$$\times\frac{[(2l_1+n-2)(2l_2+n-2)]^{1/2}(\lambda+1)}{\nabla_{n[4,7]}[\frac{1}{2}(\lambda+L_1-L_2)]!\,W_n(p_1,l_1)\,W'_n(\lambda+\nu,\bar{l}_1)}$$
$$\times\frac{(l_1l_2;\,L_1L_2)\,W_n(p_2,l_2)}{W'_n(\nu,\bar{l}_2)}$$
$$\times\sum_{x,y,z,u}\frac{(-1)^{\psi+y}2^{x-y-z}[\frac{1}{2}(\lambda+L_1-L_2)+z]!}{z!(x-y-z)!(y-z)!(L_1-\bar{l}_2-x+y+z)!}$$
$$\times\frac{(p_2-\nu+x)!(l_1+l_2+L_1-L_2+n-2+2u)!!}{(\bar{l}_2-L_2-y+z)!(\lambda+x+1)!u![\frac{1}{2}(p_2-\nu+\bar{l}_1-l_1)+y-u]!}$$
$$\times\frac{(\lambda+\nu+\bar{l}_1+n-2+2y)!!}{[\frac{1}{2}(p_2-\nu+\bar{l}_2-l_2)+x-y-u]![\frac{1}{2}(\nu-\bar{l}_2)-x+y]!}$$
$$\times\frac{(L_1-\bar{l}_2-x+2y)!(\bar{l}_2-L_2+x-2y)!}{[\frac{1}{2}(l_1+\bar{l}_1-p_2+\nu)-L_2-x+y+u]![\frac{1}{2}(l_2+\bar{l}_2-p_2-\nu)-L_2-y+u]!}$$
$$\times[(p_2-\nu+\bar{l}_1+l_1+n-2+2y)!!(p_2-\nu+\bar{l}_2+l_2+n-2+2x-2y)!!]^{-1}. \tag{7.4}$$

The bases Q and $\bar{Q}$ form a biorthogonal system. In the case $L_1-L_2>\lambda$ the system $\bar{Q}$ is overcomplete. By analogy with (3.2) the bilinear combinations of isofactors may be expressed in terms of isofactors represented by (6.9) and (7.4). In such a way the linearly dependent states of the system $\bar{Q}$ may be expanded in terms of states with index l_2 satisfying the condition (6.13).

8. Conclusions

In the present paper, the seven types of isofactors which couple the non-canonical basis states of the symmetric representations of the unitary groups have been considered as well as the seven types of labelling schemes of the repeating irreps of subgroups in the case of two-parametric irreps for the chains $SU_n \supset SO_n$ and $SU_4 \supset SU_2 \times SU_2$†.

Two principal approaches to the coupling and the labelling problem under consideration exist. One of them, which may be called *integral*, leads to a polynomial basis and to a complete system of the non-orthonormal isofactors, equivalent to bilinear combinations of orthonormal isofactors. The other approach, which may be called *differential*, leads to the generalisation of the stretched or antistretched basis and to a

† The identity of isofactors for chains $S_{N'+N''} \supset S_{N'} \times S_{N''}$ and $SU_{n'n''} \supset SU_{n'} \times SU_{n''}$ (Vanagas 1971, equation (13.25), Chen 1981, Haase and Butler 1984) allows us to apply these results for the external labelling problem in the case of the special classes of irreps of the symmetric (permutation) groups.

complete system of non-orthonormal isofactors as solutions of the boundary value problem of some equations of finite differences. The overlaps of polynomial and stretched states form a unity matrix. In such a way a biorthogonal system of bases is constructed.

The proposed variants of bases are more or less convenient in different regions of the parameters. In some regions the bases of projected or polynomial type may be overcomplete. (Luckily for us the projected basis and the basis $\bar{A}$ are never both overcomplete.) In the corresponding regions the bases of antistretched or quasi-stretched type are constructed in a more difficult way, because in such cases the additional problem of the expansion of the proper isofactors in terms of the simpler solutions of the boundary value problem has to be solved. The group theoretical meaning of the pseudoisofactors and pseudostates which appeared still awaits an explanation.

In this paper we demonstrated the high effectiveness of the different methods of the analytic continuation of the discrete functions for the coupling coefficient problem. Methods based on the relations between the complementary groups allowed us to relate the isofactors under consideration and the expansion coefficients for chains $Sp_4 \supset U_2$ and $Sp_4 \supset SU_2 \times SU_2$. Relations of the other kind allowed us to apply results obtained in the $SU_4 \supset SU_2 \times SU_2$ case for the Wigner–Racah calculus of $SU_n \supset SO_n$. The substitution group technique allowed us to relate the results obtained for mixed tensor irreps and for covariant irreps.

The metric tensor for a given non-orthogonal basis coincides with the overlaps matrix of the dual basis. The overlaps of the states of polynomial (B, $\bar{A}$, $\bar{Q}$) type are always expressed as particular cases of the corresponding non-orthonormal isofactors. The overlaps for states of projected or polynomial $\bar{A}$ type were fully expressed as triple sums, with the number of summands of each kind never exceeding the multiplicity of states of the given irrep†. A special expression for the overlaps of the stretched states was obtained as well, but the attempts to find convenient explicit expressions for overlaps of antistretched or quasistretched states of $SU_n \supset SO_n$ (for $n \geqslant 4$) were unsuccessful. Unfortunately, the methods of substitution group have certain limitations for overlaps of $SU_n \supset SO_n$ states (contrary to the $SU_3 \supset SO_3$ case (Ališauskas 1978)) because the elementary overlaps lose after substitution their group theoretical meaning.

An important question left open is the relation between the internal labelling operators for the $Sp_4 \supset U_2$, $SU_n \supset SO_n$ and $SU_4 \supset SU_2 \times SU_2$ chains in the case of two-parametric representations.

Appendix 1. On the coordination of the boundary values of isofactors

The coordinated boundary values of isofactors in region (6.3) were found by Ališauskas (1978) for $SU_3 \supset SO_3$ and by Ališauskas and Norvaišas (1980) for $SU_4 \supset SU_2 \times SU_2$. The bilinear combinations of isofactors

$$\sum_{\omega} \begin{bmatrix} (\lambda+\nu 00) & (\nu 00) & (\lambda\nu 0) \\ j_1 j_1 & j_2 j_2 & \omega, ST \end{bmatrix} \begin{bmatrix} (\lambda 00) & (0\nu 0) & (\lambda\nu 0) \\ s_1 s_1 & s_2 t_2 & \omega, ST \end{bmatrix} \tag{A1.1}$$

with the special values of parameters

$$j_1 = j_2 + T, \qquad s_2 = S - s_1 + \Delta_0, \qquad t_2 = T - s_1$$

† This condition is not always satisfied even by expressions given in appendix 3.

give a possible distribution of the boundary values in region (6.3) for isofactors which couple to the polynomial states as defined by Ahmed and Sharp (1972) and Ališauskas and Norvaišas (1980). The quantity (A1.1) was found after the examination of different coupling schemes of irreps of SU_4

$$(\lambda+\nu 00)\times(\nu 00)\times(0\nu 0) \text{ to } (\lambda 00).$$

It was easy to find a matrix, inverse of the triangular matrix formed by (A1.1), and to expand the antistretched states and corresponding isofactors in terms of polynomial states and related isofactors.

The boundary values of isofactors in region (6.8) for $l_2 < L_1 - \lambda + \Delta_0$ may be found (up to the phase factor) by means of the analytical continuation of the quantity (4.7) after substitutions grounded on the dependency between the states of the complementary groups Sp(4, R) and SO_n (see Ališauskas and Norvaišas 1980).

In order to prove the choice of phases the following analysis was needed. The quantity (6.6) with $l_1 + l_2 = L_1 + L_2$ and the quantity (6.12) with $l_1 - l_2 = L_1 - L_2$ are the expansion coefficients of the basis A in terms of the basis Q and *vice versa*, respectively. It is necessary to demonstrate that the product of those two matrices is the unity matrix. The proof is rather simple in the case $\lambda + L_2 \geqslant L_1 \geqslant \nu + \delta_0$, when the expansion matrices are triangular. Otherwise the expansion coefficients are proportional to Saalschutzian ${}_4F_3(1)$ series (see Slater 1966) and should be transformed to a more convenient form by means of the relations $(d+e+x=a+b+c)$

$$\sum_s \frac{(-1)^s(d-s)!(e-s)!(b-s-1)^{(x)}}{s!(a-s)!(b-s)!(c-s+1)!} = \frac{x!(d-a)!(e-a)!}{b!(b-x-1)!(c-e)!(c-d)!} \times \sum_{i\geqslant 0} \frac{(b-i-1)!(d-b+x-i)!(e-b+x-i)!}{(x-i)!(a-i)!(c+1-i)!}, \tag{A1.2}$$

$$\sum_s \frac{(-1)^s(d-s)!(e+s)!(b-s-1)^{(x)}}{s!(a-s)!(b-s)!(c+s+1)!} = \frac{x!(d-a)!(e+d+1)!}{b!(b-x-1)!(c-e)!(a+c+1)!} \times \sum_{j\geqslant 0} \frac{(e+j)!(b-j-1)!(d-b+x-j)!}{(x-j)!(a-j)!(b+c-x+1+j)!}. \tag{A1.3}$$

Particular cases of (A1.2) and (A1.3) with $x=0$ are equivalent to the Saalschutz summation theorem (see Slater 1966, § 2.3). The use of quasipowers $q^{(x)} = q(q-1)(q-2)\dots(q-x+1)$ allows us to join two summation intervals which appeared in (6.6) and (6.12) as consequences of (6.5) and (6.11). Equations (A1.2) and (A1.3) are proved by induction with the help of the relation

$$(b-s-x) = (1/b)[(b-x)(b-s)-sx] \qquad (b \neq 0)$$

and the Saalschutz formula.

After transformation by means of (A1.2) or (A1.3) the sum over matrix indices may be taken using the Saalschutz formula, as well as the remaining sums, and the reversibility of the transformation between the bases A and Q is proved.

Appendix 2. On the proof of expansions for overlaps (5.4) and (7.3)

The results of Católicas Norvaišas (1980) and Ališauskas (1982) allow us to write some overlaps for $SU_4 \supset SU_2 \times SU_2$ in the form

$$\left\langle \begin{matrix} E \\ k \end{matrix} \middle| \begin{matrix} E \\ k' \end{matrix} \right\rangle = \sum_{l_2, l_0} \left\langle \begin{matrix} E \\ k \end{matrix} \middle| \begin{matrix} \bar{A} \\ \underline{l}_2 \end{matrix} \right\rangle \left\langle \begin{matrix} A \\ \underline{l}_2 \end{matrix} \middle| \begin{matrix} S \\ l_0 \end{matrix} \right\rangle \left\langle \begin{matrix} B \\ l_0 \end{matrix} \middle| \begin{matrix} E \\ k \end{matrix} \right\rangle, \tag{A2.1}$$

$$\left\langle \begin{matrix} \bar{A} \\ \underline{l}'_2 \end{matrix} \middle| \begin{matrix} \bar{A} \\ \underline{l}_2 \end{matrix} \right\rangle = \sum_{l_0, k} \left\langle \begin{matrix} \bar{A} \\ \underline{l}'_2 \end{matrix} \middle| \begin{matrix} B \\ l_0 \end{matrix} \right\rangle \left\langle \begin{matrix} S \\ l_0 \end{matrix} \middle| \begin{matrix} \bar{E} \\ k \end{matrix} \right\rangle \left\langle \begin{matrix} E \\ k \end{matrix} \middle| \begin{matrix} \bar{A} \\ \underline{l}_2 \end{matrix} \right\rangle. \tag{A2.2}$$

B and S are dual bases of $SU_4 \supset SU_2 \times SU_2$ for the irrep $(\lambda\nu 0)$ as defined by Ališauskas and Norvaišas (1979); E and $\bar{E}$ are projected (Draayer 1970) and dual to such bases. The matrix $\langle S|\bar{E}\rangle$ is the inverse of the matrix $\langle B|E\rangle$ (both were found by Ališauskas and Norvaišas (1979)). The matrices $\langle A|S\rangle$ and $\langle \bar{A}|B\rangle$ are discussed in appendix 1. The matrix elements of $\langle E|\bar{A}\rangle$ can be expressed as particular cases of the quantity (5.2) in the form of special ${}_3F_2(1)$ series and transformed into a more convenient form by using the relation between the special ${}_3F_2(1)$ and Saalschutzian ${}_4F_3(1)$ series (cf Slater 1966, § 2.5). This relation may be represented in the form of a new expression for special SU_2 Clebsch–Gordan coefficients (Ališauskas 1978, 1982)

$$\begin{aligned} \begin{bmatrix} l_1 & l_2 & L \\ 0 & K & K \end{bmatrix} &= (-1)^{(l_1+l_2-L-\delta')/2}\Delta(l_1 l_2 L)(2K)! \\ &\times\left(\frac{(2L+1)(l_2-K)!(L-K)!}{(l_2+K)!(L+K)!}\right)^{1/2} \sum_x \frac{(-1)^x[\frac{1}{2}(l_1+l_2-L-\delta')+K-x]!}{(2x+\delta')!(2K-\delta'-2x)!} \\ &\times\{[\tfrac{1}{2}(l_1+l_2-L-\delta')-x]![\tfrac{1}{2}(l_1-l_2+L-\delta')-x]! \\ &\times[\tfrac{1}{2}(L-l_1+l_2+\delta')-K+x]!\}^{-1} \end{aligned} \tag{A2.3}$$

($\delta'=0$ or 1, $K \geqslant 0$, $l_1+l_2-L-\delta'$ is even) and is obtained by the complementary group technique. Here Δ is the triangle coefficient defined by (12.15) (Jucys and Bandzaitis !977) or (3.276) (Biedenharn and Louck 1981).

Equations (2.8) and (7.4) used for overlaps do not simplify considerably for low values of L_1, L_2, l_1, l_2 if values of λ, ν, μ are high. Special rearrangements of sums are useful in this domain (some examples are given in appendix 4). The overlaps of the states $\bar{Q}$ can be expressed as polynomials in λ, ν for fixed values of $L_1 - L_2$ and the multiplicity labels if they are represented in the following form:

$$\left\langle \begin{matrix} \bar{Q} \\ \bar{l}_2 \end{matrix} \middle| \begin{matrix} \bar{Q} \\ \bar{l}'_2 \end{matrix} \right\rangle = \sum_{\underline{l}_2, \underline{l}'_2} \left\langle \begin{matrix} \bar{Q} \\ \bar{l}_2 \end{matrix} \middle| \begin{matrix} A \\ \underline{l}_2 \end{matrix} \right\rangle \left\langle \begin{matrix} \bar{A} \\ \underline{l}_2 \end{matrix} \middle| \begin{matrix} \bar{A} \\ \underline{l}'_2 \end{matrix} \right\rangle \left\langle \begin{matrix} A \\ \underline{l}'_2 \end{matrix} \middle| \begin{matrix} \bar{Q} \\ \bar{l}'_2 \end{matrix} \right\rangle \tag{A2.4}$$

(after performing the transformation of the matrix elements $\langle\bar{Q}|A\rangle$ by means of (A1.3)).

A rather bulky expression (A2.4) can be convenient for analytical (especially computer based) calculations. The substitution inverse to (6.1) allows us to find the overlaps of the states B in the form of polynomials as well.

Appendix 3. Overlaps of non-canonical basis states for the $SU_3 \supset SO_3$ chain

Ališauskas (1978) obtained a simpler expression for the overlaps of the $SU_3 \supset SO_3$ non-canonical basis of Bargmann–Moshinsky B type. The proof of this formula was commented on by Ališauskas *et al* (1981). The results and methods of Ališauskas (1978) are sufficient to derive the overlaps of the $SU_3 \supset SO_3$ non-canonical bases of S, $\bar{A}$, Q type and (in the cases of the particular values of parameters) of A, Q type in the form of double sums.

Let us introduce the following notations:

$$\delta = 0 \text{ or } 1, \qquad \Delta = 0 \text{ or } 1, \qquad \bar{\Delta} = 0 \text{ or } 1,$$

so that

$$\lambda + \mu - L - \delta, \qquad \lambda - L - \Delta, \qquad \mu - L - \bar{\Delta} \tag{A3.1}$$

are even integers. It is easily seen that δ is equal to L_2 for bases B and S, while Δ is equal to L_2 for bases A, $\bar{A}$, Q, $\bar{Q}$. In all cases $L = L_1$.

The asymmetric quantities G_B, $G_{\bar{A}}$ and $G_{\bar{Q}}$ are defined as

$$\begin{aligned} G_B(l_{10}l_{20}; l'_{10}l'_{20}) &= \frac{(\mu+\delta+\Delta)!!(\lambda+L-\Delta)!!(\lambda+\mu+L+\delta)!!}{(\lambda+\mu+1)!(2L+\delta+1)!(l'_{20}-\delta-\Delta)!!} \\ &\times\left(\frac{(2l_{10}+1)!(2l_{20}+1)!!(l_{20}-\Delta-\delta)!!(2l'_{10}+1)!!(2l'_{20}+1)!}{l_{10}^{\delta}(l_{20}-\delta+\Delta-1)!!2^{l_{10}+l'_{20}}(l'_{10}-\delta)!l'^{\delta}_{20}}\right)^{1/2} \\ &\times[W'_3(\lambda, l_{10})W_3(\mu, l_{20})W_3(\lambda, l'_{10})W'_3(\mu, l'_{20})]^{-1} \\ &(l_{10}+l_{20}=l'_{10}+l'_{20}=L+\delta), \end{aligned} \tag{A3.2}$$

$$\begin{aligned} G_{\bar{A}}(\underline{l}_1\underline{l}_2; \underline{l}'_1\underline{l}'_2) &= \left(\frac{(2\underline{l}_2+1)!(L+\underline{l}_2)!2^{-\underline{l}_2-\underline{l}'_2}(2\underline{l}'_2+1)!}{(2\underline{l}_1-1)!!\underline{l}_2^{\Delta}(L+\underline{l}'_2)!(2\underline{l}'_1-1)!!\underline{l}'^{\Delta}_2}\right)^{1/2} \\ &\times\frac{(\lambda+1)!(2L-\Delta)!(\mu+\delta+\Delta)!!W_3(\lambda+\mu, \underline{l}_1)}{(\lambda+\mu+L+\delta)!!(\lambda+L-\delta)!!(\underline{l}_2-\delta-\Delta)!!(\underline{l}'_2-\Delta+\delta-1)!!} \\ &\times\frac{W'_3(\lambda+\mu, \underline{l}'_1)}{W'_3(\mu, \underline{l}_2)W_3(\mu, \underline{l}'_2)}, \qquad (\underline{l}_1-\underline{l}_2=\underline{l}'_1-\underline{l}'_2=L-\Delta), \end{aligned} \tag{A3.3}$$

$$\begin{aligned} G_{\bar{Q}}(\bar{l}_1\bar{l}_2; \bar{l}'_1\bar{l}'_2) &= [W'_3(\lambda+\mu, \bar{l}_1)W_3(\mu, \bar{l}_2)W'_3(\mu, \bar{l}'_2)]^{-1} \\ &\times\left(\frac{(2\bar{l}_1+1)!(2\bar{l}_2+1)!!(\bar{l}_2-\Delta-\delta)!!2^{\bar{l}'_1}(2\bar{l}'_1+1)!!(2\bar{l}'_2+1)!}{\bar{l}_1^{\Delta}(\bar{l}_2-\Delta+\delta-1)!!(\bar{l}'_1-\Delta)!\underline{l}'^{\Delta}_2}\right)^{1/2} \\ &\times\frac{(\lambda+1)!(\mu+\delta+\Delta)!!W_3(\lambda+\mu, \bar{l}'_1)}{(\lambda-L-\Delta-1)!!(\lambda+\mu-L+\delta-1)!!(2L+\delta+1)!(\bar{l}'_2-\Delta-\delta)!!}, \\ &(\bar{l}_1+\bar{l}_2=\bar{l}'_1+\bar{l}'_2=L+\Delta). \end{aligned} \tag{A3.4}$$

Equation (5.6) of Ališauskas (1978) may be written in the following form:

$$\begin{aligned} &\left\langle \begin{matrix} (\lambda\mu)_B \\ (l_{10}l_{20})L \end{matrix} \middle| \begin{matrix} (\lambda\mu)_B \\ (l'_{10}l'_{20})L \end{matrix} \right\rangle \\ &\quad = (-1)^{(l'_{20}-\Delta-\delta)/2}2^{\mu/2-\lambda-2}G_B(l_{10}l_{20}; l'_{10}l'_{20}) \\ &\quad \times(\lambda+L-\Delta+2)(\lambda+\mu+L+\delta+2)\sum_{l_2, z}\frac{[\frac{1}{2}(\mu-l_2)]!(L+l_2)!}{[\frac{1}{2}(l_2-\delta-\Delta)]![\frac{1}{2}(\underline{l}_2+\underline{l}'_{20})]!} \end{aligned}$$

$$\times \frac{[\frac{1}{2}(\lambda+\mu-L-l_2+\Delta)]!(l_2-\Delta+\delta-1)!!(-1)^z}{[\frac{1}{2}(\mu-l_2-l'_{10}+\Delta+\delta)]!2^{3z+3l_2/2}z![\frac{1}{2}(\mu-l_2)-z]!}$$
$$\times \frac{[\frac{1}{2}(\mu-\Delta-\delta)-z]!(\lambda+\delta+1+2z)!}{[\frac{1}{2}(l_{20}-\Delta-\delta)-z]![\frac{1}{2}(\lambda+L+l_2+\delta)+1+z]![\frac{1}{2}(\lambda-L+\Delta)+z]!}$$
$$\times \frac{(\lambda+L+\Delta+1+2z)!!}{[\frac{1}{2}(\lambda+L-l_{20}+\delta)+1+z]!}. \qquad \text{(A3.5)}$$

Equations (5.3) and (4.10) of Granted Ališauskas (1978), cf Ališauskas 1983a, equation (4.2)) give

$$\left\langle \begin{matrix} (\lambda\mu)_S \\ (l_{10}l_{20})L \end{matrix} \middle| \begin{matrix} (\lambda\mu)_S \\ (l'_{10}l'_{20})L \end{matrix} \right\rangle$$
$$= (-1)^{-(\mu-l'_{20})/2} \frac{2^{\lambda+\mu-3(\Delta+\delta)/2+1}(\mu+\Delta+\delta)}{G_B(l'_{10}l'_{20};\, l_{10}l_{20})}$$
$$\times \sum_{l_2,\, z} \frac{[\frac{1}{2}(\lambda+\mu-L-\delta)-z]![\frac{1}{2}(\mu-\Delta-\delta)-z]!(-1)^z}{[\frac{1}{2}(\lambda+\mu-L-l_2+\Delta)]!z![\frac{1}{2}(\mu-l'_{20})-z]![\frac{1}{2}(l_2-\Delta-\delta)-z]!}$$
$$\times \frac{[\frac{1}{2}(l_{20}+l_2)-1]![\frac{1}{2}(l_2-\Delta-\delta)]!2^{3l_2/2-3z}}{[\frac{1}{2}(\mu-l_2)]![\frac{1}{2}(l_{20}-\Delta-\delta-\mu+l_2)]!(l_2-\Delta+\delta-1)!!(L+l_2)!}$$
$$\times \frac{[\frac{1}{2}(\lambda+\mu+L+l_2-\Delta)-z]![\frac{1}{2}(\lambda+\mu+l'_{10}-\Delta-\delta)-z]!}{(\lambda+\mu+L-\delta+1-2z)!!(\lambda+\mu+1-\Delta-2z)!}. \qquad \text{(A3.6)}$$

Equations (5.3) and (4.9) of Ališauskas (1978) give

$$\left\langle \begin{matrix} (\lambda\mu)_{\bar{A}} \\ (\underline{l}_2)L \end{matrix} \middle| \begin{matrix} (\lambda\mu)_{\bar{A}} \\ (\underline{l}'_2)L \end{matrix} \right\rangle$$
$$= (-1)^{(\underline{l}_2-\Delta-\delta)/2} G_{\bar{A}}(\underline{l}_1\underline{l}_2;\, \underline{l}'_1\underline{l}'_2)$$
$$\times \sum_{l_0,\, z} \frac{[\frac{1}{2}(\mu-l_0)]!(l_0-\delta+\Delta-1)!!2^{\lambda+3(\mu-l_0)/2-3z}(-1)^z}{(L-l_0)![\frac{1}{2}(\underline{l}_2+l_0)]![\frac{1}{2}(\mu-\underline{l}_2-l_0+\Delta+\delta)]![\frac{1}{2}(l_0-\delta-\Delta)]!}$$
$$\times \frac{[\frac{1}{2}(\mu-\Delta-\delta)-z]![\frac{1}{2}(\lambda+\mu+L+\underline{l}'_2-\Delta)-z]!}{[\frac{1}{2}(\lambda-L+l_0-\delta)]!z![\frac{1}{2}(\mu-l_0)-z]![\frac{1}{2}(\underline{l}'_2-\Delta-\delta)-z]!}$$
$$\times \frac{[\frac{1}{2}(\lambda+\mu+L-l_0-\Delta)-z]![\frac{1}{2}(\lambda+\mu-L-\delta)-z]!}{(\lambda+\mu+L-\delta+1-2z)!!(\lambda+\mu+1-\Delta-2z)!}. \qquad \text{(A3.7)}$$

Equations (5.6), (4.9), (4.10) and (4.11) of Ališauskas (1978) allow us to obtain

$$\left\langle \begin{matrix} (\lambda\mu)_A \\ (\underline{l}_2)L \end{matrix} \middle| \begin{matrix} (\lambda\mu)_A \\ (\underline{l}'_2)L \end{matrix} \right\rangle$$
$$= (-1)^{(\mu-\underline{l}_2)/2} \frac{(\mu+\Delta+\delta)(\lambda+L-\Delta+2)(\lambda+\mu+L+\delta+2)}{2^{\lambda+3(\Delta+\delta)/2+3}G_{\bar{A}}(\underline{l}'_1\underline{l}'_2;\, \underline{l}_1\underline{l}_2)}$$
$$\times \sum_{l_2^0} \left\{ \delta_{\underline{l}_2 l_2^0} + \frac{(-1)^{(\mu-L-l_2^0+\delta+\Delta+\bar{\Delta})/2}2^{\underline{l}_2-l_2^0+1}}{(\underline{l}_2-l_2^0)[\frac{1}{2}(\mu-L-l_2^0+\delta+\Delta+\bar{\Delta})-1]!}\right.$$
$$\times \frac{[\frac{1}{2}(\lambda+\mu-L-l_2^0+\Delta)]![\frac{1}{2}(\mu-l_2^0)]!(l_2^0-\Delta+\delta-1)!!}{[\frac{1}{2}(L+\underline{l}_2-\mu-\delta-\Delta-\bar{\Delta})]![\frac{1}{2}(\lambda+\mu-\underline{l}_1)]![\frac{1}{2}(\mu-\underline{l}_2)]!}$$

$$\times\frac{[\frac{1}{2}(\underline{l}_2-\Delta-\delta)]!(L+l_2^0+\bar{\Delta}-1)!!}{(\underline{l}_2-\Delta+\delta-1)!![\frac{1}{2}(l_2^0-\Delta-\delta)]!(L+\underline{l}_2+\bar{\Delta}-1)!!}\Bigg\}$$

$$\times\sum_{l_0,z}\frac{[\frac{1}{2}(\lambda-L+l_0-\delta)]!(L-l_0)!}{[\frac{1}{2}(\mu-l_0)]!(l_0-\delta+\Delta-1)!!}$$

$$\times\frac{[\frac{1}{2}(l_0-\delta-\Delta)]![\frac{1}{2}(\underline{l}_2'+l_0)-1]!2^{3l_0/2-3z}(-1)^z}{[\frac{1}{2}(\underline{l}_2'+l_0-\mu-\Delta-\delta)]!z![\frac{1}{2}(\mu-l_2^0)-z]!}$$

$$\times\frac{[\frac{1}{2}(\mu-\Delta-\delta)-z]!(\lambda+\delta+1+2z)!}{[\frac{1}{2}(l_0-\Delta-\delta)-z]![\frac{1}{2}(\lambda-L+\Delta)+z]![\frac{1}{2}(\lambda+L-l_0+\delta)+1+z]!}$$

$$\times\frac{(\lambda+L+\Delta+1+2z)!!}{[\frac{1}{2}(\lambda+L+l_2^0+\delta)+1+z]!}. \qquad \text{(A3.8)}$$

The states of the system A as well as the overlaps are defined only for values $\underline{l}_2\geqslant\mu-L+\Delta+\bar{\Delta}+\delta$ which correspond to the linearly independent states of $\bar{A}$. In the case $L\geqslant\mu+\bar{\Delta}$ the superfluous states in $\bar{A}$ disappear and (A3.8) takes a simpler form.

The analytical continuation of equation (5.6) of Econšauskas (1978) and equation (4.2) of Econšauskas (1983a) after substitution (6.1), (6.2) and $\Delta\rightleftarrows\delta$, $l_{10}\rightarrow\bar{l}_1$, $l_{20}\rightarrow\bar{l}_2$ allows us to obtain

$$\left\langle\begin{matrix}(\lambda\mu)_{\bar{Q}}\\(\bar{l}_1\bar{l}_2)L\end{matrix}\middle|\begin{matrix}(\lambda\mu)_{\bar{Q}}\\(\bar{l}_1'\bar{l}_2')L\end{matrix}\right\rangle$$

$$=(-1)^{(\bar{l}_2'-\Delta-\delta)/2}2^{-(\lambda+\mu-L-\delta)/2}G_{\bar{Q}}(\bar{l}_1\bar{l}_2;\ \bar{l}_1'\bar{l}_2')$$

$$\times\sum_{l_0,z}\frac{[\frac{1}{2}(\mu-l_0)]!(L+l_0)!(l_0-\delta+\Delta-1)!!2^{z-l_0/2}(-1)^z}{[\frac{1}{2}(l_0-\delta-\Delta)]![\frac{1}{2}(l_0+\bar{l}_2')]![\frac{1}{2}(\mu-\bar{l}_2'-l_0+\Delta+\delta)]!}$$

$$\times\frac{[\frac{1}{2}(\mu-\Delta-\delta)-z]!(\lambda+\mu-L-l_0-\Delta-1-2z)!!}{(\lambda+L+l_0-\delta+1)!!z![\frac{1}{2}(\mu-l_0)-z]![\frac{1}{2}(\bar{l}_2-\Delta-\delta)-z]!}$$

$$\times\frac{(\lambda+\mu+L-\delta+1-2z)!!(\lambda+\mu-\bar{l}_1-2+2z)!!}{[\frac{1}{2}(\lambda+\mu-L-\delta)-z]!(\lambda+\mu-\Delta+1+2z)!}, \qquad \text{(A3.9)}$$

$$\left\langle\begin{matrix}(\lambda\mu)_Q\\(\bar{l}_1\bar{l}_2)L\end{matrix}\middle|\begin{matrix}(\lambda\mu)_Q\\(\bar{l}_1'\bar{l}_2')L\end{matrix}\right\rangle$$

$$=(-1)^{(\mu-\bar{l}_2')/2}\frac{(\mu+\delta+\Delta)(\lambda-L-\Delta+1)(\lambda+\mu-L+\delta+1)}{2^{(L-\lambda-\Delta)/2+1}G_{\bar{Q}}(\bar{l}_1'\bar{l}_2';\ \bar{l}_1\bar{l}_2)}$$

$$\times\sum_{l_0,z}\frac{[\frac{1}{2}(\bar{l}_2+l_0)-1]![\frac{1}{2}(l_0-\Delta-\delta)]!(\lambda+L+l_0-\delta+1)!!}{[\frac{1}{2}(\bar{l}_2-\mu-\delta-\Delta+l_0)]!(l_0-\delta+\Delta-1)!![\frac{1}{2}(\mu-l_0)]!}$$

$$\times\frac{2^{l_0/2+z}(-1)^z[\frac{1}{2}(\mu-\Delta-\delta)-z]!}{(L+l_0)!z![\frac{1}{2}(\mu-\bar{l}_2')-z]!}$$

$$\times\frac{[\frac{1}{2}(\lambda-L+\Delta)+z]!(\lambda+\delta+2z+1)!}{[\frac{1}{2}(l_0-\Delta-\delta)-z]!(\lambda+L+\Delta+2z+1)!!}$$

$$\times[(\lambda-L-l_0+\delta+2z+1)!!(\lambda-\bar{l}_1'+\Delta+\delta+2z+1)!!]^{-1}. \qquad \text{(A3.10)}$$

The overlap $\langle Q|Q\rangle$ may be expressed by (A3.10) only in the region $L_1\leqslant\lambda+\Delta$, in which the system $\bar{Q}$ does not include the superfluous states. Otherwise the right-hand side of (A3.10) is indefinite.

All the separate sums in (A3.5)–(A3.10) are Saalschutzian ${}_5F_4(1)$ series. Sometimes their expansion in terms of Saalschutzian ${}_4F_3(1)$ series by methods similar to those used for the proof of (A1.2), (A1.3) may be very useful because of the relations between different ${}_4F_3(1)$ series (cf appendix 4 and Slater (1966, § 4.3.5)).

Equations (A3.5) and (A3.6) have fewer summands if $l_{20} \leq l'_{20}$. Equations (A3.7), (A3.8) are more convenient for $\underline{l}'_2 \leq \underline{l}_2$ while (A3.9), (A3.10) are more convenient for $\bar{l}_2 \leq \bar{l}'_2$.

The indeterminate quantities of type $(\mu+\delta+\Delta)[\frac{1}{2}(l_{20}+l_2)-1]!$ in (A3.6), (A3.8) and (A3.10) in the case of $\mu=0$ (in this case $\Delta=\delta=l_{20}=\underline{l}_2=\bar{l}_2$) should be replaced by 2.

Appendix 4. Some special cases of isofactors and overlaps for $SU_n \supset SO_n$

Dual approaches to isofactors lead to different complementary restrictions for the intervals of summation parameters (cf Hecht and Suzuki 1983). More universal expressions exist in multiplicity-free cases with $L_1 = L_2$, namely

$$\begin{bmatrix} (p\dot{0}) & (\dot{0}q) & (\lambda\dot{0}\mu) \\ l_1 & l_2 & [L_1L_1] \end{bmatrix}$$

$$= (-1)^{\psi+(\lambda+q-L_1+l_2+n)/2}\tfrac{1}{2}[(\lambda+\mu+n-1)(2l_2+n-2)]^{1/2}$$

$$\times\delta_{l_1l_2}\begin{Bmatrix} \frac{1}{4}(p-L_1) & \frac{1}{4}(p+L_1+n-4) & \frac{1}{4}(2l_2+n-4) \\ \frac{1}{4}(q-L_1-1) & \frac{1}{4}(q+L_1+n-3) & \frac{1}{4}(\lambda+\mu+n-3) \end{Bmatrix} \qquad (A4.1a)$$

$$= \delta_{l_1l_2}\left(\frac{(\lambda+\mu+n-1)(2l_2+n-2)(p-\lambda)!(l_2-L_1)!(L_1+l_2+n-3)!}{(\lambda+q+n-1)!2^{p+\mu-L_1-l_2}}\right)^{1/2}$$

$$\times\frac{W_{n-1}(\lambda, L_1)\,W_{n-1}(\mu, L_1)}{W'_n(p, l_1)\,W'_n(q, l_2)}$$

$$\times\sum_z\frac{(-1)^{\psi+z}(\lambda+q-L_1+l_2+n-2-2z)!!}{z![\frac{1}{2}(\lambda-L_1)-z]![\frac{1}{2}(\mu-L_1)-z]!}$$

$$\times\{(l_2-L_1-2z)![\tfrac{1}{2}(p-\lambda+L_1-l_2)+z]!(2L_1+n-3+2z)!!\}^{-1}, \qquad (A4.1b)$$

$$\begin{bmatrix} (p_1\dot{0}) & (p_2\dot{0}) & (\lambda\nu\dot{0}) \\ l_1 & l_2 & [L_2L_2] \end{bmatrix}$$

$$= (-1)^{\psi+(\lambda+\nu+L_2+n)/2}\tfrac{1}{2}[(\lambda+1)/(2l_1+n-2)]^{1/2}$$

$$\times\delta_{l_1l_2}\begin{Bmatrix} \frac{1}{4}(p-L_2) & \frac{1}{4}(p_1+L_2+n-4) & \frac{1}{4}(2l_1+n-4) \\ \frac{1}{4}(p_2-L_2-1) & \frac{1}{4}(p_2+L_2+n-3) & \frac{1}{4}(\lambda-1) \end{Bmatrix} \qquad (A4.2a)$$

$$= \frac{[(\lambda+1)(2l_1+n-2)(p_1-\nu)!(p_2-\nu)!(l_1-L_2)!(l_1+L_2+n-3)!]^{1/2}}{2^{(\lambda+3\nu-L_2)/2-l_1}W'_n(p_1, l_1)\,W'_n(p_2, l_2)\,W_{n-1}(\lambda+\nu+1, L_2)}$$

$$\times\delta_{l_1l_2}W_{n-1}(\nu, L_2)\sum_z\frac{(-1)^{\psi+(\nu-L_2)/2+z}2^{2z}(p_1+p_2+n-2-2z)!!}{z![\frac{1}{2}(\nu-L_2)-z]![\frac{1}{2}(p_1-l_1)-z]![\frac{1}{2}(p_2-l_2)-z]!}$$

$$\times[(l_1-\nu+2z)!(\nu+L_2+n-3-2z)!!]^{-1}. \qquad (A4.2b)$$

Equations (2.8) and (7.4) allow us to prove that special orthonormal isofactors are equivalent to recoupling matrices of SU_2 with quatervalued parameters. Such quantities are not exotic: the Racah coefficients with quatervalued parameters (see Kildyushov 1972, Kildyushov and Kuznetzov 1973, Ališauskas 1974, Knyr *et al* 1975, Kuznetzov and Smorodinsky 1979, Suslov 1983) allow us to relate hyperspherical functions of SO_n labelled by irreps of different chains of subgroups (i.e. they are Weyl coefficients, according to Wong (1978), for symmetric irreps of SO_n and are equivalent to the recoupling matrices for the complementary group Sp(2, R) or SU (1, 1)). The most symmetric expression for $6j$-coefficients (see Jucys and Bandzaitis 1977) allowed us to obtain (A4.1b) and (A4.2b). The number of summands in those expressions is the minimum possible. Those two expressions embrace all the isofactors found by Hecht and Suzuki (1983) as well.

If $L_1 - L_2 = 1$ and $\mu - L_2$ is even, the sums in (2.8) may be rearranged in such a way (separately for $l_1 = l_2 \pm 1$) that Saalschutzian ${}_4F_3(1)$ series appear and, in their turn, they may be transformed by methods used for (A4.1) and (A4.2) (cf Slater 1966, § 4.3.5). The final expression is then

$$\begin{bmatrix} (p\dot{0}) & (\dot{0}q) & (\lambda\dot{0}\mu) \\ l_1 & l_2 & [L_1 = L_2+1, L_2] \end{bmatrix}$$
$$= \frac{W_{n-1}(\lambda, L_1)\, W_{n-1}(\mu, L_2)}{W'_n(p, l_1)\, W'_n(q, l_2)(q+l_2+n)\bar{\delta}}$$
$$\times\left(\frac{(p-\lambda)!(l_1-L_2-\bar{\delta})!(l_1+L_2-\bar{\delta}+n-2)!}{(\lambda+q+n-1)!2^{p+\mu-l_1-L_2}}\right)^{1/2}$$
$$\times\sum_z \frac{(-1)^{\psi+z}(p+\mu-L_2+l_1+n-2-2z)!!}{z![\frac{1}{2}(\lambda-L_1)-z]![\frac{1}{2}(\mu-L_2)-z]!}$$
$$\times[(l_2-L_2+1)^{\bar{\delta}}(p-\lambda+1)(2L_2+n-1+2z)-(-1)^{\bar{\delta}}(L_2+l_2+n-1)^{\bar{\delta}}$$
$$\times(\mu+L_2+n-1)(l_1-L_1+1-2z)]\{[\tfrac{1}{2}(p-\lambda+L_1-l_1)+z]!$$
$$\times(l_1-L_1+1-2z)!(2L_2+n-1+2z)!!\}^{-1}. \qquad \text{(A4.3)}$$

(Here $\bar{\delta} = \delta_{l_1,\, l_2+1}$.)

A similar transformation may be fulfilled in the non-multiplicity-free $L_1 - L_2 = 2$ case, but the final result is too bulky to be given here. Let us present only the overlaps obtained in this way from (2.8) ($L_1 = L_2 + 2$).

$$\left\langle \begin{matrix} (\lambda\dot{0}\mu)_B \\ (L_1L_2)[L_1L_2] \end{matrix} \middle| \begin{matrix} (\lambda\dot{0}\mu)_B \\ (L_1L_2)[L_1L_2] \end{matrix} \right\rangle$$
$$= \frac{H_n(\lambda\mu\,;\, L_2+2,\, L_2)}{(\lambda+L_2+n)}$$
$$\times[(\lambda+L_2+n-1)(\lambda+\mu+n-1)(2L_2+n-1)-\mu-L_2-n+1], \qquad \text{(A4.4)}$$

$$\left\langle \begin{matrix} (\lambda\dot{0}\mu)_B \\ (L_1L_2)[L_1L_2] \end{matrix} \middle| \begin{matrix} (\lambda\dot{0}\mu)_B \\ (L_2L_1)[L_1L_2] \end{matrix} \right\rangle$$
$$= -H_n(\lambda\mu\,;\, L_2+2,\, L_2)(\lambda+\mu+2L_2+2n-2)$$
$$\times\left(\frac{(\lambda-L_2)(\mu-L_2)}{(\lambda+L_2+n)(\mu+L_2+n)}\right)^{1/2}. \qquad \text{(A4.5)}$$

Here

$$H_n(\lambda\mu;L_2+2,L_2) = \frac{(\lambda+\mu+n-4)!!(\lambda+L_2+n-3)!!(\mu+L_2+n-3)!!(2L_2+n-2)!!}{(\lambda+\mu+n-3)!!(\lambda+L_2+n-2)!!(\mu+L_2+n-2)!!(2L_2+n-1)!!}. \tag{A4.6}$$

The remaining overlap of B states with $l_{10} = L_2$, $l_{20} = L_1 = L_2+2$ may be found by means of the permutations (3.6).

The following expression for the special isofactor is of importance (Norvaišas and註išauskas 1974):

$$\begin{bmatrix} (p_1\dot{0}) & (p_2\dot{0}) & (p_1+p_2\dot{0}) \\ l_1 & l_2 & [L_1\dot{0}] \end{bmatrix} = \left(\frac{p_1!p_2!L_1!(2l_1+n-2)(2l_2+n-2)}{2^{n-3}(n-4)!!(L_1+n-3)!(p_1+p_2)!}\right)^{1/2} \times(-1)^{\psi}\frac{W_n(p_1+p_2,L_1)\nabla_{n[4,5,6,7]}(l_1l_2;L_10)}{W_n(p_1,l_1)W_n(p_2,l_2)}. \tag{A4.7}$$

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